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REGULARITY OF THE SOLUTIONS FOR ELLIPTIC PROBLEMS ON NONSMOOTH DOMAINS IN $\mathbb{R}^3$

PART II: REGULARITY IN NEIGHBORHOODS OF EDGES

by

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PART II: REGULARITY IN NEIGHBORHOODS OF EDGES

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Abstract

This paper is the second in a series of three devoted to the analysis of regularity of solutions of elliptic problems on nonsmooth domains in $\mathbb{R}^3$. The present paper concentrates on the regularity of solution of Poisson equation in neighborhoods of edges of a polyhedral domain in the frame of the weighted Sobolev spaces and countably normed spaces.

These results can be generalized to elliptic problems arising from mechanics and engineering, for instance, the elasticity problem on polyhedral domains. Hence, the results are not only important to comprehensively understand the qualitative and quantitative aspects of the behaviours of the solution and its derivatives of all orders in neighbourhoods of edges, but also essential to design an effective computation and analyze the optimal convergence of the finite elements solutions for these problems.

Keywords: Regularity in neighborhood of edges, Polyhedral domains, Weighted spaces, Countably normed spaces.

AMS(MOS) Subject Classification: 35A20, 35B65, 35D10, 35G15, 35J05.

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1. INTRODUCTION

In engineering applications the domains of the problems under consideration are often unions and intersections of simple geometrical objects such as cylinders, balls, cones, etc.. The unions and intersections of these simple objects yield edges and vertices. It is well known that the singularities of the solutions occur near the edges and vertices. The singularities make the computation for these problems on the domains with edges and vertices extremely inefficient and inaccurate. Hence precise description of the singularity is not only significant for the regularity theory of partial differential equations on nonsmooth domain, but also extremely important for the construction of effective numerical approximation methods.

This paper is the second of a series devoted to the analysis of regularity of solutions of elliptic equations on nonsmooth domains in $\mathbb{R}^3$, and it will concentrate on the regularity in neighborhoohs of edges of a polyhedral domains. The typical description of the edge singularity is the asymptotic expansion of singular functions (see [7,8,9,11,12,13,17,19,20,22])

$$u(x) = \sum_{j=1}^{J}\sum_{s=0}^{S}\sum_{t=0}^{T} C_{jst}(x_3)\psi_{jst}(\theta)r^{\alpha_j+t}(\ln r)^s + u_0$$

where (r, θ, x_3) are the local cylinderical coordinates, C_{jst} and ψ_{jst} are analytic in x_3 (except vertices) and in θ, respectively. Recently the classical weighted Sobolev spaces $\mathbf{W}_\beta^k$ and $\mathbf{V}_\beta^k$ with Kontrat'ev- and Maz'ya-type weights were used to analysing the regularity of high-order derivatives of solutions (see [18,21,23]). As indicated in previous paper [15], these approaches do not sufficiently characterize the behaviour of solutions near the edges. The solutions $u(x)$ in the edge-neighborhood is analytic except at the edge, and their derivative of order $k \geq 1$ may grow rapidly as x tends to the edge and as k increases. The regularity results in terms of the asymptotic expansions and the classical weighted Sobolev spaces are unable to reflect these natures of regularity in the edge-neighborhood. The classical weighted Sobolev spaces $\mathbf{W}_\beta^k$ and $\mathbf{V}_\beta^k$ with $0 < \beta < 1$ are suitable only for the regularity of lower-order derivatives of the solution, but not for higher-order derivatives, for instance, $k > 2$ if the elliptic equation is of the second order.

In this paper we will analyze the regularity of solution in the edge-neighborhoods in the frame of the weighted Sobolev spaces and countably normed spaces with dynamical weights. The theory of these spaces on the edge-neighborhoods has been well established in previous paper [15]. The regularity results in terms of these spaces will provide us with the complete qualitative and quantitative informations of the derivatives of solution at all orders and will lead us to the exponential convergence of the approximation by properly selected piecewise polynomial spaces (see [8,9,14,16]).

Although the regularity results for problems in vertex-neighborhoods of polygonal domains are similar to those for problems in edge-neighborhoods of polyhedral domains (see [2,3]), it is worth indicating that there are differences on the substances and approaches. We will elaborate the substantial differences in Section 4.

The notations and definitions of various spaces will be quoted in Section 2 from the previous paper [15]. The Section 3 deals with the existence and uniqueness

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In this paper we will analyze the regularity of solution in the edge-neighborhoohs in the frame of the weighted Sobolev spaces and countably normed spaces with dynamical weights. The theory of these spaces on the edge-neighborhoods has been well established in previous paper [15]. The regularity results in terms of these spaces will provide us with the complete qualitative and quantitative informations of the derivatives of solution at all orders and will lead us to the exponential convergence of the approximation by properly selected piecewise polynomial spaces (see [8,9,14,16]).

Although the regularity results for problems in vertex-neighborhoods of polygonal domains are similar to those for problems in edge-neighborhoods of polyhedral domains (see [2,3]), it is worth indicating that there are differences on the substances and approaches. We will elaborate the substantial differences in Section 4.

The notations and definitions of various spaces will be quoted in Section 2 from the previous paper [15]. The Section 3 deals with the existence and uniqueness

of weak solution of Poisson equation on polyhedral domain with data given in the corresponding weighted Sobolev spaces. The main part of the paper is Section 4 in which the regularity of solutions in the edge-neighborhoods will be derived in the frame of the dynamical weighted Sobolev spaces and countably normed spaces. These regularity results for Poisson equation can be generalized to linear elliptic equation and system of equations without substantial difficulties.

2. PRELIMINARY

We shall quote the notations and definition of the spaces which were introduced in Part I and will be used in this paper.

Let Ω be a polyhedral domain in $\mathbb{R}^3$ as shown in Fig. 2.1, and let $\Gamma_i, i \in \mathcal{I} = \{1,2,3,\cdots,I\}$ be the faces (open), Λ_{ij} be the edge which is the intersection of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$, and $A_m, m \in \mathcal{M} = \{1,2,\cdots M\}$ be the vertices of Ω. By $\mathcal{I}_m$ we denote a subset $\{j \in I | A_m \in \bar{\Gamma}_j.\}$ of I for $m \in M$. Let $\mathcal{L} = \{ij | i,j \in I, \bar{\Gamma}_i \cap \bar{\Gamma}_j = \Lambda_{ij}\}$, and let $\mathcal{L}_m$ denote a subset of $\mathcal{L}$ such that $\mathcal{L}_m = \{ij \in \mathcal{L} | A_m \in \Lambda_{ij}\}$. We denote by ω_{ij} the interior angle between Γ_i and Γ_j for $ij \in \mathcal{L}$. Let $\Gamma^0 = \bigcup_{i \in D} \Gamma_i$ and $\Gamma^1 = \bigcup_{i \in \mathcal{N}} \Gamma_i$ where $\mathcal{D}$ is a subset of I and $\mathcal{N} = \mathcal{I} \backslash \mathcal{D}$. Further, let $\mathcal{D}_m = \mathcal{D} \bigcap \mathcal{I}_m$ and $\mathcal{N}_m = \mathcal{N} \bigcap \mathcal{I}_m$ for $m \in \mathcal{M}$.

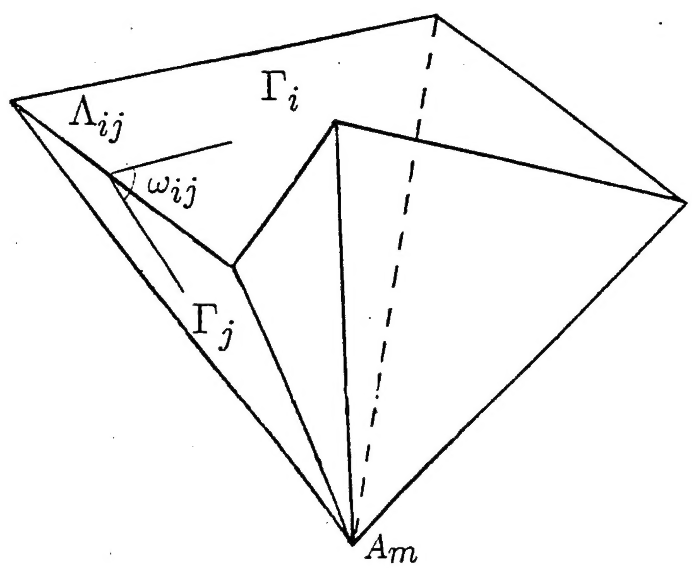

Fig. 2.1 Polyhedral domain Ω

For precise description of the regularity of solutions of elliptic problems in polyhedral domains. We have decomposed in [15] the domain into neighborhoods of edges and vertices as shown in Fig. 2.2 and introduced the weighted Sobolev spaces and weighted continous function spaces, and the countably normed spaces in these neighborhoods. The structures of these space have been fully studied in [15].

Assume that the edge Λ_{ij} lies in the x_3-axis and $\Lambda_{ij} = \{(0,0,x_3) | \quad a < x_3 < b\}$. Then a neighborhood of Λ_{ij} is defined as

$$\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) = \{x \in \Omega | 0 < r = dist(x, \Lambda_{ij}) < \varepsilon_{ij}, \quad a + \delta_{ij} < x_3 < b - \delta_{ij}\}$$

$0 < \varepsilon_{ij}, \delta_{ij} < 1$ are such that $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij}) \bigcap \bar{\Gamma}_\ell = \phi$ for $\ell \neq i, j$.

By $\mathcal{O}_{\delta_m}(A_m)$ we denote a neighborhood of the vertex A_m

$$\mathcal{O}_{\delta_m}(A_m) = \{x \in \Omega \mid 0 < \rho = dist(x, A_m) < \delta_m\}$$

Here A_m is assumed the origin and $\rho = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. $\delta_m \in (0,1)$ is selected such that $\mathcal{O}_{\delta_m}(A_m) \bigcap \bar{\Gamma}_\ell = \phi$ for any $\ell \bar{\in} \mathcal{L}_m$.

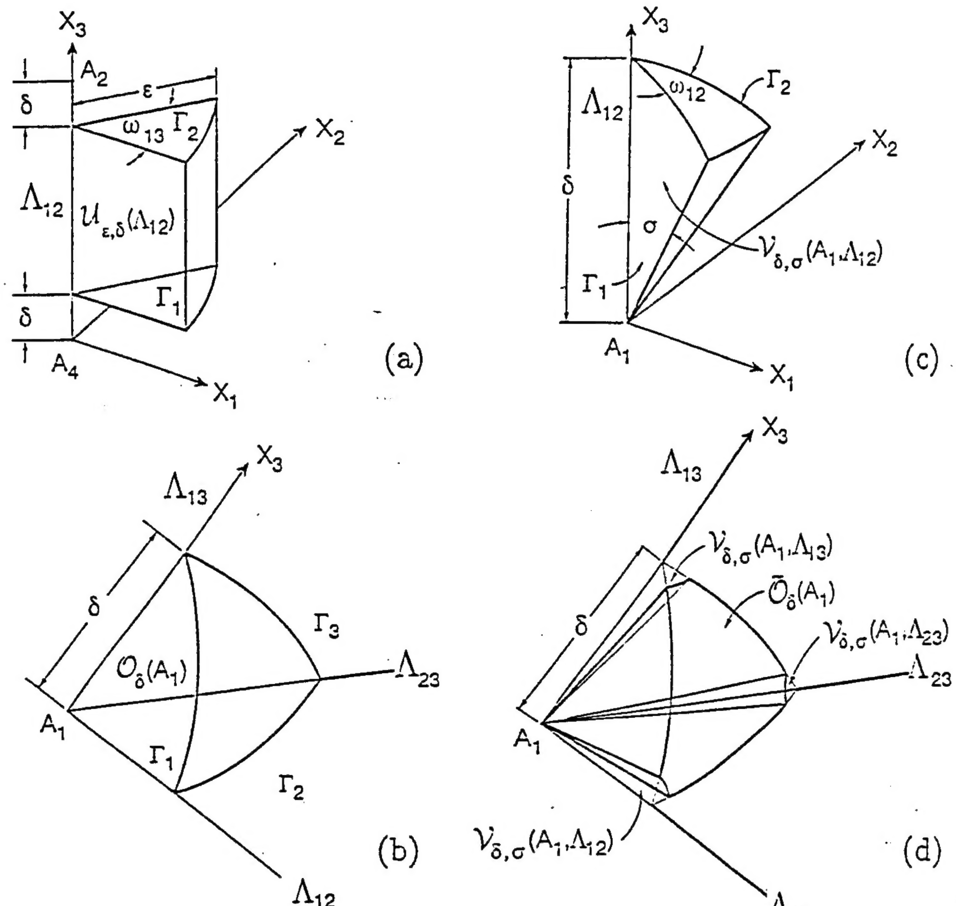

Fig. 2.2 Neighborhoods of edges and vertices
(a) the neighborhood $\mathcal{U}_{\epsilon_{ij},\delta_{ij}}(\Lambda_{ij})$; (b) the neighborhood $\mathcal{O}_{\delta_m}(A_m)$;
(c) the neighborhood $\mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij})$; (d) the inner neighborhood $\tilde{\mathcal{O}}_{\delta_m}(A_m)$.

$\mathcal{O}_{\delta_m}(A_m)$ is further decomposed into an inner neighborhood of vertex and several neighborhoods of vertex-edge. For $ij \in \mathcal{L}_m$ we define a neighborhood of the vertex A_m and edge Λ_{ij}

$$\mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij}) = \{x \in \mathcal{O}_{\delta_m}(A_m) | 0 < \phi < \sigma_{ij}\}$$

where $\phi = \phi(x)$ is the angle between Λ_{ij} and the radia from A_m to x. We assume further that Λ_{ij} lies in the positive x_3-axis. Then $\sin\phi = \sqrt{x_1^2 + x_2^2}/\rho$. $\delta_{ij} \in (0,1)$ is such that $\bar{\mathcal{V}}_{\delta_m,\delta_{ij}}(A_m, \Lambda_{ij}) \bigcap \bar{\mathcal{V}}_{\delta_m,\sigma_{k\ell}}(A_m, \Lambda_{k,\ell}) = A_m$ for all $ij \in \mathcal{L}_m$ and $k\ell \in \mathcal{L}_m$, $ij \neq k\ell$. The inner-neighborhood $\tilde{\mathcal{O}}_{\delta_m}(A_m)$ is defined as

$$\tilde{\mathcal{O}}_{\delta_m}(A_m) = \mathcal{O}_{\delta_m}(A_m) \setminus \bigcup_{ij \in L_m} \mathcal{V}_{\delta_m,\sigma_{ij}}(A_m, \Lambda_{ij}).$$

For the sake of simplicity, we shall write $\mathcal{U}_{ij}$ or $\mathcal{U}(\Lambda_{ij})$ $\mathcal{V}_{m,ij}$ or $\mathcal{V}(A_m, \Lambda_{ij})$, $\tilde{\mathcal{O}}_m$ or $\tilde{\mathcal{O}}(A_m)$ instead of $\mathcal{U}_{\varepsilon_{ij},\delta_{ij}}(\Lambda_{ij})$, $\mathcal{V}_{\delta_m,\delta_{ij}}(A_m, \Lambda_{ij})$ and $\tilde{\mathcal{O}}_{\delta_m}(A_m)$.

By $\mathbf{H}^k(\Omega), k \geq 0$ integers we denote the usual Sobolev spaces on Ω with the norm

$$\|u\|^2_{\mathbf{H}^k(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|^2_{\mathbf{L}^2(\Omega)}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D^\alpha u = u_{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}$. As usual we write $\mathbf{H}^0(\Omega) = \mathbf{L}^2(\Omega)$, $\mathbf{H}^1_0(\Omega) = \{u \in \mathbf{H}^1(\Omega) | u = 0 \text{ on } \Gamma^0\}$, and $|u|^2_{\mathbf{H}^k(\Omega)} = \sum_{|\alpha|=k} \|D^2 u\|^2_{\mathbf{L}^2(\Omega)}$ (semi-norm), and $|D^k u|^2 = \sum_{|\alpha|=k} |D^\alpha u|^2$.

The weighted Sobolev spaces are defined individually in the neighborhood of edges and vertices.

For $x \in \tilde{\mathcal{O}}_m$, $\beta_m \in (0, 1/2)$ and integers $\ell \geq 0$ we define the weight function.

$$\Phi^{\alpha,\ell}_{\beta_m}(x) = \begin{cases} \rho^{\beta_m + |\alpha| - \ell} & \text{for } |\alpha| \geq l, \\ 1 & \text{for } |\alpha| < \ell. \end{cases}$$

and weighted Sobolev spaces with integers $k \geq \ell$

$$\mathbf{H}^{k,\ell}_{\beta_m}\left(\tilde{\mathcal{O}}_m\right) = \left\{ u \mid \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)} = \sum_{0 \leq |\alpha| \leq k} \left\| \Phi^{\alpha,\ell}_{\beta_m} D^\alpha u \right\|^2_{\mathbf{L}^2(\tilde{\mathcal{O}}_m)} < \infty \right\}$$

We next construct a weight function in $\mathcal{V}_{m,ij} = \mathcal{V}(A_m, \Lambda_{ij})$, with integers $\ell \geq 0$, $\beta_{m,ij} = (\beta_m, \beta_{ij})$, $\beta_m \in (0, 1/2)$ and $\beta_{ij} \in (0, 1)$, as follows:

$$\Phi^{\alpha,\ell}_{\beta_{m,ij}}(x) = \begin{cases} \rho^{\beta_m + |\alpha| - \ell} (\sin\phi)^{\beta_{ij} + \alpha_1 + \alpha_2 - \ell} & \text{for } \ell \leq \alpha_1 + \alpha_2 \leq |\alpha|, \\ \rho^{\beta_m + |\alpha| - \ell} & \text{for } \alpha_1 + \alpha_2 < \ell \leq |\alpha|, \\ 1 & \text{for } |\alpha| < \ell. \end{cases}$$

Then we introduce the weighted Sobolev spaces over $\mathcal{V}_{m,ij}$ with integer $k \geq \ell$

$$\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij}) = \left\{ u \mid \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} = \sum_{\sigma \leq |\alpha| \leq k} \left\| \Phi^{\alpha,\ell}_{\beta_{m,ij}} D^\alpha u \right\|^2_{\mathbf{L}^2(\mathcal{V}_{m,ij})} < \infty \right\}$$

A weight function in the edge-neighborhood $\mathcal{U}_{ij} = \mathcal{U}(\Lambda_{ij})$ is defined as

$$\Phi^{\alpha,\ell}_{\beta_{ij}}(x) = \begin{cases} r^{\beta_{ij} + \alpha_1 + \alpha_2 - \ell} & \text{for } \alpha_1 + \alpha_2 \geq \ell, \\ 1 & \text{for } \alpha_1 + \alpha_2 < \ell, \end{cases}$$

with an integer $\ell \geq 0$ and $\beta_{ij} \in (0, 1)$. Then the weighted Sobolev space $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ is given by

$$\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) = \left\{ u \mid \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_{ij}}} = \sum_{|\alpha|\le k} \left\| \Phi^{\alpha,\ell}_{\beta_{ij}} D^\alpha u \right\|^2_{\mathbf{L}^2(\mathcal{U}_{ij})} < \infty \right\}.$$

Here β_{ij} coincides with that for the space $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{V}_{m,ij})$. As usual $\left|D^\ell u\right|^2 = \sum_{|\alpha|=\ell} |D^\alpha u|^2$ and

$$|u|^2_{\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})} = \sum_{|\alpha|=k} \left\| \Phi^{\alpha,\ell}_{\beta_{ij}} D^\alpha u \right\|^2_{\mathbf{L}^2(\mathcal{U}_{ij})}.$$

Let $\beta = (\beta_m, m \in \mathcal{M}, \beta_{ij}, ij \in \mathcal{L})$ with $\beta_m \in (0,1/2)$, $\beta_{ij} \in (0,1)$ be a multi-index. Then the space $H^{k,\ell}_\beta(\Omega)$ denotes the set of functions such that their restrictions on $\mathcal{U}_{ij}, \tilde{\mathcal{O}}_m, \mathcal{V}_{m,ij}$ and Ω_0 belong to $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$, $\mathbf{H}^{k,\ell}_{\beta_m}\left(\tilde{\mathcal{O}}_m\right)$ $\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ and $\mathbf{H}^k(\Omega_0)$, respectively, for all $ij \in \mathcal{L}$ and $m \in \mathcal{M}$, and

$$\begin{aligned} \|u\|^2_{\mathbf{H}^{k,\ell}_\beta(\Omega)} = & \sum_{ij\in\mathcal{L}} \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})} + \sum_{m\in\mathcal{M}} \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_m}(\tilde{\mathcal{O}}_m)} + \\ & \sum_{m\in\mathcal{M}} \sum_{ij\in\mathcal{L}_m} \|u\|^2_{\mathbf{H}^{k,\ell}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} + \|u\|^2_{\mathbf{H}^k(\Omega_0)} \end{aligned}$$

The regularity of solutions in the edge-neighborhood $\mathcal{U}_{ij}$ will be given in terms of countably normed spaces with weighted Sobolev norm

$$\mathbf{B}^\ell_{\beta_{ij}}(\mathcal{U}_{ij}) = \left\{ u \,|\, u \in \mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) \quad \textit{for all } k \ge \ell, \left\| \Phi^{k,\ell}_{\beta_{ij}}, D^\alpha u \right\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \le C d^\alpha \alpha! \right\}$$

and countably normed space with the weighted $\mathbf{C}^k$−norm

$$\begin{aligned} \mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij}) = \ & \Big\{ u \in C^0(\bar{\mathcal{U}}_{ij}) \ \mid \ r^{\beta_{ij}+|\alpha|-1} D^\alpha u \in \mathbf{C}^0(\bar{\mathcal{U}}_{ij}) \text{ for } \alpha \text{ with } |\alpha| = |k| \ge 2, \\ & \|r^{\beta_{ij}+|\alpha|-1} D^\alpha (u(x) - u(0,0,x_3))\|_{\mathbf{C}^0(\bar{\mathcal{U}}_{ij})} \le C d^\alpha \alpha! \\ & \|\frac{d^k}{dx_3^k} u(0,0,x_3)\|_{\mathbf{C}^0(\bar{I}_{\delta_{ij}})} \le C d_3^k k! \text{ for } k \ge 0 \Big\} \end{aligned}$$

Hereafter $I_{\delta_{ij}} = (a + \delta_{ij}, b - \delta_{ij})$, $d^\alpha = d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3}$, $\alpha! = \alpha_1!\alpha_2!\alpha_3!$, $C \ge 1$, and $d_i \ge 1$ are independent of α.

The relations between the spaces $\mathbf{B}^2_{\beta_{ij}}(\mathcal{U}_{ij})$ and $\mathbf{C}^2_{\beta_{ij}}(\mathcal{U}_{ij})$ has been discussed in [15] from which we quote a theorem.

Theorem 2.1 $\mathbf{B}^2_{\beta_{ij}}(\mathcal{U}_{ij}) \subset \mathbf{C}^2_{\beta_{ij}}\left(\bar{\mathcal{U}}_{ij}\right) \subset \mathbf{B}^2_{\beta_{ij}+\varepsilon}(\mathcal{U}_{ij}), \varepsilon > 0$ arbitrary. □

It is convenient to use cylindrical coordinates (r, θ, x_3) with respect to the edge Λ_{ij} when we analyze the regularities of solutions in the edge-neighborhood $\mathcal{U}_{ij}$. Hence we introduced weighed Sobolev spaces in the cylindrical coordinates

$$\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) = \left\{ u \mid \|u\|^2_{\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})} = \sum_{|\alpha|\le k} \left\| \Phi^{\alpha,\ell}_{\beta_{ij}} r^{-\alpha_2} \mathcal{D}^\alpha u \right\|^2_{\mathbf{L}^2(\mathcal{U}_{ij})} < \infty \right\}$$

and the countably weighted Sobolev spaces

$$\mathcal{B}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) = \left\{ u \in \mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij}) \text{ for all } k \geq \ell, \ \left\| \Phi^{\alpha,\ell}_{\beta_{ij}} r^{-\alpha_2} \mathcal{D}^\alpha u \right\|_{\mathbf{L}^2(\mathcal{U}_{ij})} \leq C d^\alpha \alpha! \right\}$$

where $\mathcal{D}^\alpha u = u_{r^{\alpha_1}\theta^{\alpha_2}x_3^{\alpha_3}}$.

The following theorem, which gives us the relations between spaces in Cartesian coordinates and those in the cylindrical coordinates, has been proved in [15] .

Theorem 2.2 For $\ell \leq 2$ the spaces $\mathbf{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ and $\mathcal{H}^{k,\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ are equivalent, and the space $\mathbf{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$ is equivalent to the space $\mathcal{B}^{\ell}_{\beta_{ij}}(\mathcal{U}_{ij})$. □

3. WEAK SOLUTION OF POISSON EQUATION IN POLYHEDRAL DOMAIN

Consider the Poisson equation in polyhedral domain Ω

$$\left\{ \begin{array}{l} -\Delta u = f, \\ u|_{\Gamma^0} = g^0 \\ \frac{\partial u}{\partial n}|_{\Gamma^1} = g^1 \end{array} \right. \tag{3.1}$$

with $f \in L_\beta(\Omega), g^\ell = G^\ell|_{\Gamma^\ell}$ and $G^\ell \in H^{2-\ell,2-\ell}_\beta(\Omega), \ell = 0, 1$

Lemma 3.1 If $f \in \mathbf{L}_{\beta_m}\left(\tilde{O}_m\right), 0 < \beta_m < 1/2$ and $v \in \mathbf{H}^1\left(\tilde{O}_m\right)$, then

$$\left| \int_{\tilde{O}_m} f v dx \right| \leq C \|f\|_{\mathbf{L}_\beta(\tilde{O}_m)} \|v\|_{\mathbf{H}^1(\tilde{O}_m)}. \tag{3.2}$$

Proof. By Schwartz's inequality

$$\begin{aligned} \left| \int_{\tilde{O}_m} f v dx \right| \leq \ & C \|f\|_{\mathbf{L}_{\beta_m}(\tilde{O}_m)} \left\| \rho^{-\beta_m} v \right\|_{\mathbf{L}^2(\tilde{O}_m)} \\ & \leq C \|f\|_{\mathbf{L}_{\beta_m}(\tilde{O}_m)} \|v\|_{\mathbf{L}^4(\tilde{O}_m)} \\ & \leq C \|f\|_{\mathbf{L}_{\beta_m}(\tilde{O}_m)} \|v\|_{\mathbf{H}^1(\tilde{O}_m)}. \end{aligned}$$

Here we used the fact that $\beta_m \in (0, 1/2)$ and the imbedding result of Sobolev spaces (see [1]). □

Lemma 3.2 If $f \in \mathbf{L}_{\beta_{ij}}(\mathcal{U}_{ij}), 0 < \beta_{ij} < 1$ and $v \in \mathbf{H}^1(\mathcal{U}_{ij})$, then

$$\left| \int_{\mathcal{U}_{ij}} f v dx \right| \leq C \|f\|_{\mathbf{L}_{\beta_{ij}}(\mathcal{U}_{ij})} \|v\|_{\mathbf{H}^1(\mathcal{U}_{ij})}. \tag{3.3}$$

Proof. By Schwartz's inequality

$$\left| \int_{\mathcal{U}_{ij}} f v dx \right| \leq \|f\|_{\mathbf{L}_{\beta_{ij}}(\mathcal{U}_{ij})} \left\| r^{-\beta_{ij}} v \right\|_{\mathbf{L}^2(\mathcal{U}_{ij})}. \tag{3.4}$$

By Lemma 5.1 of [Part 1]

$$\begin{aligned}\left|\int_{\mathcal{U}_{ij}} r^{-2\beta_{ij}} |v|^2\, dx\right| &= \left|\int_{\mathcal{U}_{ij}} r^{2(1-\beta_{ij})-2} |v|^2\, dx\right| \\ &\leq C\int_{\mathcal{U}_{ij}} r^{2(1-\beta_{ij})} \left(\left|D^1 v\right|^2 + |v|^2\right) dx\end{aligned}$$

which together with (3.4) yields (3.3). □

Lemma 3.3 If $f \in \mathbf{L}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$, $\beta_{m,ij} = (\beta_m, \beta_{ij})$ with $\beta_m \in (0, 1/2)$, and $\beta_{ij} \in (0,1)$, and $v \in \mathbf{H}^1(\mathcal{V}_{m,ij})$, then

$$\left|\int_{\mathcal{V}_{m,ij}} fv dx\right| \leq C \|f\|_{\mathbf{L}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} \|v\|_{\mathbf{H}^1(\mathcal{V}_{m,ij})}. \tag{3.5}$$

Proof. By Schwartz's inequality we have

$$\left|\int_{\mathcal{V}_{m,ij}} fv dx\right| \leq \|f\|_{\mathbf{L}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} \left\|\rho^{-\beta_m} (\sin\phi)^{-\beta_{ij}}\, v\right\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})}. \tag{3.6}$$

Let $\tilde{\beta}_m = 1/2 - \beta_m$ and $\tilde{\beta}_{ij} = 1 - \beta_{ij}$ and $\tilde{\beta}_{m,ij} = \left(\tilde{\beta}_m, \tilde{\beta}_{ij}\right)$. Then by Lemma 4.2 of [15]

$$\begin{aligned}\left\|\rho^{-\beta_m} (\sin\phi)^{-\beta_{ij}}\, v\right\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &= \int_{\mathcal{V}_{m,ij}} \rho^{2\tilde{\beta}_m - 2} (\sin\phi)^{2\tilde{\beta}_{ij}-2} |\rho^{1/2} v|^2 dx \\ &\leq C\|\rho^{1/2} v\|_{\mathbf{H}^{1,1}_{\tilde{\beta}_{m,ij}}(\mathcal{V}_{m,ij})^2} \\ &\leq C\|\rho^{1/2} v\|_{\mathbf{H}^1(\mathcal{V}_{m,ij})}.\end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned}\left\|\rho^{-1/2} v\right\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} &\leq \|\rho^{-1}\|_{\mathbf{L}^2(\mathcal{V}_{m,ij})} \|v\|_{\mathbf{L}^4(\mathcal{V}_{m,ij})} \\ &\leq C\|v\|_{\mathbf{H}^1(V_{m,ij})}.\end{aligned} \tag{3.8}$$

Then (3.5) follows from (3.6)-(3.8). □

Combining Lemma 3.1-3.3 we have

Theorem 3.1 If $f \in \mathbf{L}_\beta(\Omega)$, then $f \in \left(\mathbf{H}^1(\Omega)\right)'$, and

$$\|f\|_{\left(\mathbf{H}^1(\Omega)\right)'} \leq C\|f\|_{\mathbf{L}_\beta(\Omega)}.$$

□

Lemma 3.4 Let $G \in \mathbf{H}^{1,1}_{\beta_{ij}}(\mathcal{U}_{ij})$. Then for $v \in \mathbf{H}^1(\mathcal{U}_{ij})$

$$\left|\int_{\Gamma_i \bigcap \partial\mathcal{U}_{ij}} G\, v ds\right| \leq C \left\|G\right\|_{\mathbf{H}^{1,1}_{\beta_{ij}}(\mathcal{U}_{ij})} \left\|v\right\|_{\mathbf{H}^1(\mathcal{U}_{ij})}. \tag{3.9}$$

Proof. Let $\mathcal{U}_{ij} = Q_\varepsilon \times I_{\delta_{ij}}$, $\quad Q_\varepsilon = \left\{ (x_1, x_2) \;\middle|\; 0 < \sqrt{x_1^2 + x_2^2} = r < \varepsilon \right\}$ and $I_{\delta_{ij}} = (a + \delta_{ij}, b - \delta_{ij})$, and let $\gamma = \bar{Q}_\varepsilon \bigcap \Gamma_i$. By $\mathbf{H}^{1,1}_{\beta_{ij}}(Q_\epsilon)$ we denote the weighted Sobolev space over Q_ε

$$\mathbf{H}^{1,1}_{\beta_{ij}}(Q_\epsilon) = \left\{ w(x_1, x_2) \;\middle|\; \|w\|^2_{\mathbf{H}^{1,1}_{\beta_{ij}}(Q_\epsilon)} = \|w\|^2_{\mathbf{L}^2(Q_\epsilon)} + \sum_{|\alpha'|=1} \left\| r^{\beta_{ij}} D^{\alpha'} w \right\|^2_{\mathbf{L}^2(Q_\epsilon)} < \infty \right\}.$$

Then for almost every $x_3 \in I_{\delta_{ij}}$, $\quad G \in \mathbf{H}^{1,1}_{\beta_{ij}}(Q_\varepsilon)$ and $v \in \mathbf{H}^1(Q_\varepsilon)$. By Lemma 2.11 of [2] we have

$$\int_\gamma |G| \; |v| \, ds \le C \|G\|_{\mathbf{H}^{1,1}_{\beta_{ij}}(Q_\epsilon)} \|v\|_{\mathbf{H}^1(Q_\epsilon)} \tag{3.10}$$

where C is a constant independent of x_3, integrating (3.10) in x_3 over $I_{\delta_{ij}}$ we obtain (3.9). □

Lemma 3.5 If $G \in \mathbf{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})$ with $\beta_m \in (0, 1/2)$ and $\beta_{ij} \in (0,1)$, then for $v \in H^1(\mathcal{V}_{m,ij})$

$$\left| \int_{\Gamma_i \bigcap \partial \mathcal{V}_{m,ij}} G v ds \right| \le C \|G\|_{\mathbf{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})} \|v\|_{\mathbf{H}^1(\mathcal{V}_{m,ij})}. \tag{3.11}$$

Proof. Let $S^\sigma_{ij} = \{(\phi, \theta) | \, 0 < \phi < \sigma, 0 < \theta < \omega_{ij}\}$ and $I_\delta = (0, \delta_m)$. Then $\mathcal{V}_{m,ij} = S^\sigma_{ij} \times I_{\delta_m}$. We may assume that Γ_i is in the $x_1 - x_3$ plane. Let $\gamma = \partial S^\sigma_{ij} \bigcap \Gamma_i = \{(\phi, \theta) | \, 0 < \phi < \sigma, \theta = 0\}$. Then by Lemma 2.11 of [2]

$$\int_\gamma |G| \, |v| \, ds \le C \|G\|_{\mathbf{H}^{1,1}_{\beta_{ij}}} \|v\|_{\mathbf{H}^1(S^\sigma_{ij})} \tag{3.12}$$

where $\mathbf{H}^1\left(S^\sigma_{ij}\right)$ and $\mathbf{H}^{1,1}_{\beta_{ij}}\left(S^\sigma_{ij}\right)$ are the Sobolev and weighted Sobolev spaces over S^σ_{ij} namely

$$\|v\|^2_{\mathbf{H}^1(S^\sigma_{ij})} = \int_{S^\sigma_{ij}} \left(|v|^2 + |v_\phi|^2 + \frac{1}{\phi^2} |v_\theta|^2 \right) \phi d\phi d\theta$$

and

$$\|G\|^2_{\mathbf{H}^{1,1}_{\beta_{ij}}(S^\sigma_{ij})} = \int_{S^\sigma_{ij}} \left\{ |G|^2 + (\sin\phi)^{2\beta_{ij}} \left(|G_\phi|^2 + \left| \frac{1}{\phi} G_\theta \right|^2 \right) \right\} \phi d\phi d\theta.$$

Multiplying (3.12) with ρ and integrating it in ρ over I_δ we get

$$\int_{\gamma \times I_{\delta_m}} |G| \, |v| \, \rho d\rho d\phi \le C \int_{I_{\delta m}} \rho \|G\|_{\mathbf{H}^{1,1}_{\beta_{ij}}(S^\sigma_{ij})} \|v\|_{\mathbf{H}^1(S^\sigma_{ij})} d\rho \tag{3.13}$$

by Schwartz's inequality

$$\leq C\left(\int_{I_{\delta_m}}\rho^{2\beta_m}\|G\|^2_{\mathbf{H}^{1,1}_{\beta_{ij}}(S^\sigma_{ij})}\,d\rho\right)^{1/2}\left(\int_{I_{\delta_m}}\rho^{2(1-\beta_m)}\|v\|^2_{\mathbf{H}^1(S^\sigma_{ij})}\,d\rho\right)^{1/2}.$$

By Lemma 4.1 of [15] we have

$$\begin{aligned}
&\int_{I_\delta}\rho^{2\beta_m}\|G\|^2_{\mathbf{H}^{1,1}_{\beta_{ij}}(S^\sigma_{ij})}\,d\rho\\
&\leq C\int_{\mathcal{V}_{m,ij}}\left\{\rho^{2\beta_m-2}|G|^2+\rho^{2\beta_m}(\sin\phi)^{2\beta_{ij}}(|\frac{1}{\rho}G_\phi|^2+|\frac{1}{\rho\sin\phi}G_\theta|^2)\right\}dx \qquad (3.14)\\
&\leq C\|G\|^2_{\mathbf{H}^{1,1}_{\beta_{m,ij}}(\mathcal{V}_{m,ij})}.
\end{aligned}$$

Analogously, using the fact that $\beta_m\in(0,1/2)$ and the imbedding theorem of Sobolev space

$$\begin{aligned}
&\int_{I_\delta}\rho^{2(1-\beta_m)}\|v\|^2_{\mathbf{H}^1(S^\sigma_{ij})}\,d\rho\\
&\leq C\int_{\mathcal{V}_{m,ij}}\left\{\rho^{-2\beta_m|v|^2}+\rho^{2(1-\beta_m)}\left(\left|\frac{1}{\rho}v_\phi\right|^2+\left|\frac{1}{\rho\sin\phi}v_\theta\right|^2\right)\right\}dx \qquad (3.15)\\
&\leq C\|v\|^2_{\mathbf{H}^1(\mathcal{V}_{m,ij})}.
\end{aligned}$$

□

The combination of (3.13)-(3.15) leads to (3.11).

Lemma 3.6 Let $G\in\mathbf{H}^{1,1}_{\beta_m}\left(\tilde{\mathcal{O}}_m\right)$. Then for $v\in\mathbf{H}^1\left(\tilde{\mathcal{O}}_m\right)$ and $i\in\mathcal{I}_m$

$$|\int_{\Gamma_i\cap\tilde{\mathcal{O}}_m}GvdS|\leq C\|G\|_{\mathbf{H}^{1,1}_\beta(\tilde{\mathcal{O}}_m)}\|v\|_{\mathbf{H}^1(\tilde{\mathcal{O}}_m)}. \qquad (3.16)$$

Proof. Let $\tilde{S}=S\setminus\cup_{ij\in\mathcal{L}_m}S^\sigma_{ij}$ where S is the intersection of Ω and the infinite polyhedral which coincides with Ω in the vertex neighborhood $\mathcal{O}(A_m)$ and S^σ_{ij} were defined in the proof for the previous lemma. Then $\tilde{\mathcal{O}}_m=\tilde{S}\times I_{\delta_m}$. The proof of (3.16) is similar to that of (3.11) except that

$$\int_\gamma|G\|v|\,ds\leq C\|G\|_{\mathbf{H}^1(\tilde{S})}\|v\|_{\mathbf{H}^1(\tilde{S})}$$

instead of (3.12), and

$$\int_{I_\delta}\rho^{2\beta_m}\|G\|^2_{\mathbf{H}^1(\tilde{S})}\,d\rho\leq C\|G\|^2_{\mathbf{H}^{1,1}_{\beta_m}(\tilde{\mathcal{O}}_m)}$$

instead of (3.14), and

$$\int_{I_\delta}\rho^{2(1-\beta_m)}\|v\|^2_{\mathbf{H}^1(\tilde{S})}\,d\rho\leq\|v\|_{\mathbf{H}^1(\tilde{\mathcal{O}}_m)}$$

instead of (3.15). $\square$

Lemma 3.4-3.6 lead us to

Thoerem 3.2 If $G \in \mathbf{H}_{\beta}^{1,1}(\Omega)$, then for $v \in \mathbf{H}^1(\Omega)$

$$\left|\int_{\partial\Omega} Gvds\right| \leq C \|G\|_{\mathbf{H}_{\beta}^{1,1}(\Omega)} \|v\|_{\mathbf{H}^1(\Omega)}. \tag{3.17}$$

$\square$

We are now ready to prove the theorem of the existence and uniqueness of the weak solution for the problem (3.1) with f and G^{ℓ} given in the corresponding weighted Sobolev spaces.

Theorem 3.3 Let Ω be a polyhedra in $\mathbb{R}^3$, $f \in \mathbf{L}_{\beta}(\Omega)$, $g^{\ell} = G^{\ell}|_{\Gamma^{\ell}}$ and $G^{\ell} \in \mathbf{H}_{\beta}^{2-\ell,2-\ell}(\Omega), \ell = 0, 1$ with $\beta_m \in (0, 1/2)$ and $\beta_{ij} \in (0,1)$ for all $m \in \mathcal{M}$ and $ij \in \mathcal{L}$. Then the problem (3.1) has a unique solution $u \in \mathbf{H}^1(\Omega)$ (weak sense) such that $u - G^0 \in \mathbf{H}_0^1(\Omega)$, and

$$\|u\|_{\mathbf{H}^1(\Omega)} \leq C\left(\|f\|_{\mathbf{L}_{\beta}(\Omega)} + \sum_{\ell=0,1} \left\|G^{\ell}\right\|_{\mathbf{H}_{\beta}^{2-\ell,2-\ell}(\Omega)}\right). \tag{3.18}$$

Proof. We may assume that $g^0 = 0$. The bilinear form on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$ is defined as

$$B(u,v) = \int_{\Omega} \nabla u \cdot \nabla v dx.$$

Due to Theorem 3.1 and 3.2

$$F(v) = \int_{\Omega} fvdx + \int_{\Gamma^1} g^1 vds$$

defines a linear functional on $\mathbf{H}^1(\Omega)$, and

$$\|F\|_{\left(\mathbf{H}^1(\Omega)\right)'} \leq C\left(\|f\|_{\mathbf{L}_{\beta}(\Omega)} + \left\|G^1\right\|_{\mathbf{H}_{\beta}^{1,1}(\Omega)}\right).$$

By Lax-Milgram theorem there exists a unique solution $u \in \mathbf{H}_0^1(\Omega)$ for the variational equation of the problem (3.1)

$$B(u,v) = F(v), \quad \forall v \in \mathbf{H}_0^1(\Omega), \tag{3.19}$$

and

$$\|u\|_{\mathbf{H}^1(\Omega)} \leq C\left(\|f\|_{\mathbf{L}_{\beta}(\Omega)} + \left\|G^1\right\|_{\mathbf{H}_{\beta}^{1,1}(\Omega)}\right)$$

which is (3.15) with $g^0 = 0$. For general case that $g^0 \neq 0$ (3.16) can be proven easily.
$\square$

Remark 3.1 If $|\Gamma^0| = 0$ and

$$\int_\Omega f dx + \int_{\Gamma^1} g^1 ds = 0, \tag{3.20}$$

then Theorem 3.3 holds in the quotient space module a constant. □

4. REGULARITY IN NEIGHBORHOODS OF EDGES

We shall make further investigation on the regularities of the solution of (3.1) in neighborhoods of the edges in frame of the weighted Sobolev spaces and countably normed spaces. We concentrate ourself on a neighborhood $\mathcal{U}_{12}$ of the edge Λ_{12}. As assumed in previous sections, Λ_{12} lies in the x_3-axis and $\mathcal{U}_{12} = \{(r,\theta,x_3) \mid (r,\theta) \in Q_\varepsilon,$ $x_3 \in I_\delta\}$ with $Q_\varepsilon = \{(r,\theta) \mid 0 < r < \varepsilon, 0 < \theta < \omega\}$ and $I_\delta = (-1+\delta, 1-\delta)$. (r,θ,x_3) are the cylindrical coordinates with respect to Λ_{12}. We further assume that $\Gamma_1 \subset \Gamma^0, \Gamma_2 \subset \Gamma^1$. For sake of simplicity we shall write $\mathcal{U} = \mathcal{U}_{\varepsilon,\delta} = \mathcal{U}_{12} = \mathcal{U}_{\varepsilon,\delta}(\Lambda_{12}), Q = Q_\varepsilon$ etc.. As in Section 2 we denote $D^\alpha u = D^{\alpha'} u_{x_3^{\alpha_3}} = u_{x_1^{\alpha_1} x_i^{\alpha_2} x_3^{\alpha_3}}$ and $\mathcal{D}^\alpha u = \mathcal{D}^{\alpha'} u_{x_3^{\alpha_3}}$ $= u_{r^{\alpha_1}\theta^{\alpha_2}x_3^{\alpha_3}}$ with $\alpha = (\alpha', \alpha_3) = (\alpha_1, \alpha_2, \alpha_3)$ and $|\alpha| = |\alpha'| + \alpha_3 = \alpha_1 + \alpha_2 + \alpha_3$. We shall write $\mathbf{H}^{k,\ell}_{\beta_{12}}(\mathcal{U}) = \mathbf{H}^{k,\ell}_{\beta_{12}}(\mathcal{U}_{12})$ and $\mathbf{B}^{\ell}_{\beta_{12}}(\mathcal{U}) = \mathbf{B}^{\ell}_{\beta_{12}}(\mathcal{U}_{12})$, etc.

4.1 Regularity of high-order derivatives with respect to the direction along the edges

Lemma 4.1 Let $\mathrm{T} = \{(x_1,x_3) \mid x_1 \in (0,\varepsilon), x_3 \in I_\delta\}$ and $G \in \mathbf{H}^{1/2}(S)$. Suppose that $v \in \mathbf{H}^1(T)$ and $v = 0$ for $x_1 = \varepsilon$ or $x_3 = \pm(1-\delta)$. Then

$$\left|\int_T G \Delta_h v ds\right| \le C \|G\|_{\mathbf{H}^{1/2}(\mathrm{T})} \|v\|_{\mathbf{H}^{1/2}(\mathrm{T})} \tag{4.1}$$

where $\Delta_h v = \frac{1}{h}(v(x_1, x_3+h) - v(x_1,x_3))$, C is a constant independent of G and v.
Proof. First we extend G and v into $\tilde{\mathrm{T}} = (-\varepsilon,\varepsilon) \times I_\delta$ by symmetric manner with respect to x_1-axis. The extended functions are denoted by $\tilde{G}$ and $\tilde{v}$. Then $\tilde{G} \in \mathbf{H}^{1/2}(\mathrm{T})$ and $\tilde{v} \in \mathbf{H}^1_0(\mathrm{T})$. Further we extend $\tilde{v}$ in whole plane by zero extension outside T, and extend $\tilde{G}$ in the plane as well. Then the $\mathbf{H}^{1/2}$-norm of $\tilde{G}$ and $\mathbf{H}^{1/2}_{0,0}$-norm of $\tilde{v}$ are preserved, and $\Delta_h v$ is well defined. Let $\hat{G}$ and $\hat{v}$ denote the Fourier transformation of $\tilde{G}(\xi,\eta)$ and $\tilde{v}(\xi,\eta)$. The equivalent norms of $\tilde{G}$ and $\tilde{v}$ in $\mathbf{H}^{1/2}(\mathbb{R}^2)$ are defined as

$$\|\tilde{G}\|_{\mathbf{H}^{1/2}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} (1+\xi^2+\eta^2)^{1/2} \left|\hat{G}\right|^2 d\xi d\eta\right)^{1/2}$$

and

$$\|\tilde{v}\|_{\mathbf{H}^{1/2}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \left(1+\xi^2+\eta^2\right)^{1/2} |\hat{v}|^2 d\xi d\eta\right)^{1/2}.$$

Then

$$
(4.2)\qquad \begin{aligned}
\int_{\mathrm{T}} G\Delta_h v dS &= \tfrac{1}{2}\int_{\tilde{T}} \tilde{G}\Delta_h \tilde{v} dS = \frac{1}{2}\int_{\mathbb{R}^2} \tilde{G}\Delta_h \tilde{v} dS \\
&= \tfrac{1}{2}\int_{\mathbb{R}^2} \tilde{G}\tilde{v}\frac{e^{-ih\xi}-1}{h} d\xi d\eta
\end{aligned}
$$

For $|h\xi| \le 1$

$$
\left|\frac{e^{-i\xi h}-1}{h}\right| \cdot \frac{1}{(1+\xi^2+\eta^2)^{1/2}} \le \frac{|\xi|}{(1+\xi^2+\eta^2)^{1/2}} \le 1
$$

and for $|h\xi| > 1$

$$
\left|\frac{e^{-i\xi h}-1}{h}\right| \frac{1}{(1+\xi^2+\eta^2)^{1/2}} \le \frac{2}{|\xi h|} < 2.
$$

Hence we have by Schwartz's inequality

(4.3)

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2} \hat{G}\hat{v}\frac{e^{-i\xi h}-1}{h} d\xi d\eta\right| \\
&\le 2\left(\int_{\mathbb{R}^2} \left|\hat{G}\right|^2 \left(1+\xi^2+\eta^2\right)^{1/2} d\xi d\eta\right)^{1/2} \left(\int_{\mathbb{R}^2} |v|^2 \left(1+\xi^2+\eta^2\right)^{1/2} d\xi d\eta\right)^{1/2} \\
&\le 2\left\|\hat{G}\right\|_{\mathbf{H}^{1/2}\left(\mathbb{R}^2\right)} \|\hat{v}\|_{\mathbf{H}^{1/2}\left(\mathbb{R}^2\right)} \\
&\le C\|\tilde{G}\|_{\mathbf{H}^{1/2}(\tilde{\mathrm{T}})} \|\tilde{v}\|_{\mathbf{H}^{1/2}(\tilde{\mathrm{T}})} \\
&\le 4C\,\|G\|_{\mathbf{H}^{1/2}(\mathrm{T})} \|v\|_{\mathbf{H}^{1/2}(\mathrm{T})}
\end{aligned}
$$

Then (4.1) follows from (4.2) and (4.3) at once. □

Select $\varepsilon' \in (\varepsilon, 1)$ and $\delta' \in (0, \delta)$ and let $\mathcal{U}' = \mathcal{U}_{\varepsilon',\delta'} = \mathcal{U}_{\varepsilon',\delta'}(\Lambda_{12}) \subset \Omega$. Then $\mathcal{U}' \supset \mathcal{U} = \mathcal{U}_{\varepsilon_\delta}$, and we have the following theorems.

Theorem 4.1 Let $u \in \mathbf{H}^1(\Omega)$ be the weak solution of the problem (3.1) with $f \in \mathbf{L}_\beta(\Omega)$ and $G^\ell \in \mathbf{H}_\beta^{2-\ell,2-\ell}(\Omega), \ell = 0, 1$.

(i) If $f \in \mathbf{L}^2(\mathcal{U}')$ and $G^\ell \in \mathbf{H}^{2-\ell}(\mathcal{U}')$, $\ell = 0, 1$, then $u_{x_3} \in \mathbf{H}^1(\mathcal{U})$, and

$$
(4.4)\qquad \begin{aligned}
&\|u_{x_3}\|_{\mathbf{H}^1(\mathcal{U})} \\
&\le C_0\left\{\|f\|_{\mathbf{L}^2(\mathcal{U}')} + \sum_{\ell=0,1}\left\|G^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}')} + M_0\|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}')}\right\} \\
&\le C_0\left\{\|f\|_{\mathbf{L}^2(\mathcal{U}')} + \|f\|_{\mathbf{L}_\beta(\Omega)} + \sum_{\ell=0,1}\left(\left\|G^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}')} + \left\|G^\ell\right\|_{\mathbf{H}_\beta^{2-\ell,2-\ell}(\Omega)}\right)\right\}.
\end{aligned}
$$

(ii) If $f_{x_3} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$ and $G_{x_3}^\ell \in \mathbf{H}_{\beta_{12}}^{2-\ell,2-\ell}(\mathcal{U}')$, then $u_{x_3} \in \mathbf{H}^1(\mathcal{U})$, and

(4.5)

$$
\begin{aligned}
&\|u_{x_3}\|_{\mathbf{H}^1(\mathcal{U})}\\
&\leq C_0\left\{\sum_{m=0,1}(M_0^{1-m}\|f_{x_3^m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}+\sum_{\ell=0,1}\|G^\ell_{x_3^m}\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}')})+M_0\|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}')}\right\}\\
&\leq C_0\left\{\|f\|_{\mathbf{L}_\beta(\Omega)}+\|f_{x_3}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}+\sum_{\ell=0,1}\left\|G^\ell\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta}(\Omega)}\left\|G^\ell_{x_3}\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}')}\right\}.
\end{aligned}
$$

(iii) If $f=f_1+f_2$ with $f_1\in\mathbf{L}^2(\mathcal{U}')$, and $f_{2,x_3}\in\mathbf{L}_{\beta_{12}}(\mathcal{U}')$, $G^\ell=G_1^\ell+G_2^\ell$ with $G_1^\ell\in\mathbf{H}^{2-\ell}(\mathcal{U}')$ and $G^l_{2,x_3}\in\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')$, then $u_{x_3}\in\mathbf{H}^1(\mathcal{U})$ and

$$
\begin{aligned}
&\|u_{x_3}\|_{\mathbf{H}^1(\mathcal{U})}\\
&\leq C_0\left\{\|f_1\|_{\mathbf{L}^2(\mathcal{U}')}+\sum_{\ell=0,1}\left\|G_1^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}')}+\sum_{m=0,1}\left\|f_{2,x_3^m}\right\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}\right.\\
&\quad\left.+\sum_{\ell=0,1}\left\|G^\ell_{2,x_3^m}\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}')}+M_0\left\|u_{x_3}\right\|_{\mathbf{L}^2(\mathcal{U}')}\right\}\qquad(4.6)\\
&\leq C_0\left\{\|f\|_{\mathbf{L}_\beta(\Omega)}+\sum_{\ell=0,1}\left\|G^\ell\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta}(\Omega)}+\|f_1\|_{\mathbf{L}^2(\mathcal{U}')}+\|f_{2,x_1}\|_{\mathbf{L}^2_{\beta_{12}}(\mathcal{U}')}\right.\\
&\quad\left.+\sum_{\ell=0,1}\left(\left\|G_1^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}')}+\left\|G^\ell_{2,x_3}\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}')}\right)\right\}.
\end{aligned}
$$

Here C_0 is a constant independent of x_3, and

$$
M_0=\max\left\{\frac{1}{\Delta\delta},\frac{1}{\Delta\varepsilon}\right\},\quad \Delta\delta=\delta-\delta',\Delta\varepsilon=\varepsilon'-\varepsilon. \tag{4.7}
$$

Proof. First we assume that $G^0=0$. Let $\Delta_h u=\frac{1}{h}(u(x+he_3)-u(x))$ with $h\in(0,\delta)$ and $e_3=(0,0,1)$, and let $\tilde{\mathbf{H}}^1(\Omega)=\left\{u\in\mathbf{H}^1(\Omega)|u=0\ \ for\ \ x\in\Omega\backslash\mathcal{U}'\right\}$. By the standard argument of difference quotient (see, e.g. [10]), we have for any $w\in\tilde{\mathbf{H}}^1(\Omega)$

$$
\begin{aligned}
\int_{\mathcal{U}'}\nabla(\Delta_h u)\nabla w dx&=\int_\Omega(\Delta_h\nabla u)\cdot\nabla w dx \qquad (4.8)\\
&=-\int_\Omega\nabla u\cdot\nabla(\Delta_{-h}w)\,dx
\end{aligned}
$$

by (3.19)

$$
\begin{aligned}
&=-\int_\Omega f(\Delta_{-h}w)\,dx-\int_{\Gamma^1}G^1(\Delta_{-h}w)\,dS\\
&=-\int_{\mathcal{U}'}f(\Delta_{-h}w)\,dx-\int_{\Gamma_2\bigcap\partial\mathcal{U}'}G^1(\Delta_{-h}w)\,dS.
\end{aligned}
$$

In the case (i), $f\in\mathbf{L}^2(\mathcal{U}')$ and $G^1\in\mathbf{H}^1(\mathcal{U}')$. We have by Schwartz's inequality

$$\left|\int_{\mathcal{U}'} f\left(\Delta_{-h} w\right) dv\right| \leq C\,\|f\|_{\mathbf{L}^2(\mathcal{U}')}\,\|\Delta_{-h} w\|_{\mathbf{L}^2(\mathcal{U}')} \tag{4.9}$$

by Lemma 7.2.3 of [10]

$$\leq C\|f\|_{\mathbf{L}^2(\mathcal{U}')}\|w_{x_3}\|_{\mathbf{L}^2(\mathcal{U}')}.$$

By Lemma 4.1 we have

$$\left|\int_{\Gamma_2\bigcap\partial\mathcal{U}'} G^1\left(\Delta_{-h} w\right) dS\right| \leq C\left\|G^1\right\|_{\mathbf{H}^{1/2}(\Gamma_2\bigcap\partial\mathcal{U}')}\,\|w\|_{\mathbf{H}^{1/2}(\Gamma_2\bigcap\partial\mathcal{U}')} \tag{4.10}$$

by the imbedding inequality of Sobolev space (see [1])

$$\begin{aligned}&\leq C\,\|G^1\|_{\mathbf{H}^1(\mathcal{U}')}\,\|w\|_{\mathbf{H}^1(\mathcal{U}')}\\ &\leq C\,\|G^1\|_{\mathbf{H}^1(\mathcal{U}')}\,\|\nabla w\|_{\mathbf{L}^2(\mathcal{U}')}.\end{aligned}$$

Here we used the inequality

$$\|w\|_{\mathbf{H}^1(\mathcal{U}')} \leq C\,\|\nabla w\|_{\mathbf{L}^2(\mathcal{U}')}. \tag{4.11}$$

Combining (4.8)-(4.10) we have

$$\left|\int_{\mathcal{U}'} \nabla\left(\Delta_h u\right)\cdot\nabla w dx\right| \leq C\left(\|f\|_{\mathbf{L}^2(\mathcal{U}')}+\left\|G^1\right\|_{\mathbf{H}^1(\mathcal{U}')}\right)\|\nabla w\|_{\mathbf{L}^2(\mathcal{U}')}. \tag{4.12}$$

Let $\varphi_1(x_3)$ and $\varphi_2(r)$ be C^∞ cut-off functions such that $0 \leq \varphi_1(x_3), \varphi_2(r) \leq 1$, and

$$\varphi_1(x_3)=\begin{cases}1, & \text{for } |x_3|\leq 1-\delta'\\ 0, & \text{for } |x_3|>1-\delta\end{cases}, \quad \phi_2(r)=\begin{cases}1, & \text{for } r\leq\varepsilon\\ 0, & \text{for } r>\varepsilon'.\end{cases} \tag{4.13}$$

Set $\eta=\varphi_1(x_3)\varphi_2(r)$ and $w=\eta^2\Delta_h u$. Then

$$\nabla\left(\Delta_h u\right)\cdot\nabla w=\nabla\left(\Delta_h u\right)\nabla\left(\eta^2\Delta_h u\right)=\left|\eta\nabla\left(\Delta_h u\right)\right|^2+2\eta\left(\Delta_h u\right)\nabla\eta\cdot\nabla\left(\Delta_h u\right)$$

which together with (4.12) yields

$$\begin{aligned}&\int_{\mathcal{U}'}\left|\eta\nabla\left(\Delta_h u\right)\right|^2 dx\\ &\leq C\left(\|f\|_{\mathbf{L}^2(\mathcal{U}')}+\|G^1\|_{\mathbf{H}^1(\mathcal{U}')}\right)\|\nabla\left(\eta^2\Delta_h u\right)\|_{\mathbf{L}^2(\mathcal{U}')}\\ &\quad+2\,\|\eta\nabla\left(\Delta_h u\right)\|_{\mathbf{L}^2(\mathcal{U}')}\cdot\|\left(\Delta_h u\right)\nabla\eta\|_{\mathbf{L}^2(\mathcal{U}')}\\ &\leq C\left(\|f\|_{\mathbf{L}^2(\mathcal{U}')}+\|G^1\|_{H^1(\mathcal{U}')}\right)\left(\|\eta\nabla\left(\Delta_h u\right)\|_{\mathbf{L}^2(\mathcal{U}')}+\|\left(\Delta_h u\right)\nabla\eta\|_{\mathbf{L}^2(\mathcal{U}')}\right)\\ &\quad+2\,\|\left(\Delta_h u\right)\nabla\eta\|_{\mathbf{L}^2(\mathcal{U}')}\cdot\|\eta\nabla\left(\Delta_h u\right)\|_{\mathbf{L}^2(\mathcal{U}')}\end{aligned} \tag{4.14}$$

Let $A(f,G^1) = C\left(\|f\|_{\mathbf{L}^2(\mathcal{U}')} + \|G^1\|_{\mathbf{H}^1(\mathcal{U}')}\right)$. Then we have

$$\|(\nabla_h u)\nabla\eta\|_{\mathbf{L}^2(\mathcal{U}')}\|\eta\nabla(\Delta_h u)\|_{\mathbf{L}^2(\mathcal{U}')} \le \frac{1}{8}\|\eta\nabla(\Delta_h u)\|^2_{\mathbf{L}^2(\mathcal{U}')} + 2\|(\Delta_h u)\nabla\eta\|^2_{\mathbf{L}^2(\mathcal{U}')},$$

$$\mathrm{A}\left(f,G^1\right)\|\eta\nabla(\Delta_h u)\|_{\mathbf{L}^2(\mathcal{U}')} \le 2\left|\mathrm{A}\left(f,G^1\right)\right|^2 + \frac{1}{8}\|\eta\nabla(\Delta_h u)\|^2_{\mathbf{L}^2(\mathcal{U}')},$$

$$\mathrm{A}\left(f,G^1\right)\|(\Delta_h u)\nabla\eta\|_{\mathbf{L}^2(\mathcal{U}')} \le \frac{1}{2}\left|\mathrm{A}\left(f,G^1\right)\right|^2 + \frac{1}{2}\|(\Delta_h u)\nabla\eta\|^2_{\mathbf{L}^2(\mathcal{U}')}.$$

Substitution of these inequalities into (4.14) gives

$$\|\eta\nabla(\Delta_h u)\|^2_{\mathbf{L}^2(\mathcal{U}')} \le \tilde{C}\left(\|f\|^2_{\mathbf{L}^2(\mathcal{U}')} + \left\|G^1\right\|^2_{\mathbf{H}^1(\mathcal{U}')} + \|(\Delta_h u)\nabla\eta\|^2_{\mathbf{L}^2(\mathcal{U}')}\right).$$

Note that $\eta = 1$ in $\mathcal{U}$, $\quad |\nabla\eta| \le CM_0$ with $M_0 = \max\left(\frac{1}{\Delta\varepsilon}, \frac{1}{\Delta\delta}\right)$. Then by Lemma 7.23 of [10] we obtain (4.4) for $G^0 = 0$.

In the case (ii), $f_{x_3} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}'), G^{\ell}_{x_3} \in \mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}'), \ell = 0,1$. Then for any $w \in \mathbf{H}^1_0(\Omega)$

$$\left|\int_\Omega f\Delta_{-h} w dx\right| = \left|\int_{\mathcal{U}'}(\Delta_h f)\, w dx\right| \tag{4.15}$$

by Schwartz's inequality

$$\le \|\Delta_h f\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}\left\|r^{-\beta_{12}} w\right\|_{\mathbf{L}^2(\mathcal{U}')}$$

by Lemma 5.1 of [15]

$$\le C\|\Delta_h f\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}\|w\|_{\mathbf{H}^1(\mathcal{U}')}$$

by Lemma 7.23 of [10] and (4.11)

$$\le C\|f_{x_3}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}\|\nabla w\|_{\mathbf{L}^2(\mathcal{U}')}.$$

Due to Lemma 3.2

$$\begin{aligned} \left|\int_{\Gamma_2\bigcap\partial\mathcal{U}'} G^1(\Delta_{-h} w)\, dS\right| &\le \left|\int_{\Gamma_2\bigcap\partial\mathcal{U}'}(\Delta_h G^1)\, w dS\right| \\ &\le C\left\|\Delta_h G^1\right\|_{\mathbf{H}^{1,1}_{\beta_{12}}(\mathcal{U}')}\|w\|_{\mathbf{H}^1(\mathcal{U}')} \end{aligned} \tag{4.16}$$

by Lemma 7.23 of [10] and (4.11)

$$\le C\|G_{x_3}\|_{\mathbf{H}^{1,1}_{\beta_{12}}(\mathcal{U}')}\|\nabla w\|_{\mathbf{L}^2(\mathcal{U}')}.$$

Combining this with (4.8), (4.15) and (4.16) we obtain

$$(4.17)\qquad \int_{\mathcal{U}'} \nabla(\Delta_h u)\cdot\nabla w dx \le C\left(\|f_{x_3}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} + \left\|G^1_{x_3}\right\|_{\mathbf{H}^{1,1}_{\beta_{12}}(\mathcal{U}')}\right)\|\nabla w\|_{\mathbf{L}^2(\mathcal{U}')}.$$

Then the proof of (4.5) is the same as that for (4.4) except that the inequality (4.17) is used instead of (4.12).

(4.6) for $G^0 = 0$ in the case (iii) is obtained by combining (4.4) and (4.5).

We now prove (4.4) and (4.6) is general, i.e. $G^0|_{\Gamma_1} \neq 0$. Let $v = u - G^0$. Then v satisfies

$$\begin{cases} -\Delta v = f + \Delta_{12}G^\circ + G^0_{x_3^2} = \tilde{f}, \\ v|_{\Gamma_1} = 0, \\ \dfrac{\partial v}{\partial n}|_{\Gamma_2} = \left(G^1 + \dfrac{\partial G^0}{\partial n}\right)|_{\Gamma_2} = \tilde{G}^1, \end{cases}$$

where $\Delta_{12} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. If $f \in \mathbf{L}^2(\mathcal{U}')$, $\quad G^\ell \in \mathbf{H}^{2-\ell}(\mathcal{U}')$, then $\tilde{f} \in \mathbf{L}^2(\mathcal{U}')$ and $\tilde{G}^1 \in \mathbf{H}^1(\mathcal{U}')$. Applying (4.4) with $\tilde{G}^0 = 0$ we obtain (4.4) in general. If $f_{x_3} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$, $\quad G^\ell_{x_3} \in \mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}')$, $\quad \ell = 0,1$ then $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$ with $\tilde{f}_1 = G^0_{x_3^2} \in \mathbf{L}^2(\mathcal{U}')$, $\quad \tilde{f}_2 = (f + \Delta_{12}G^0) \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$ and $\tilde{f}_{2,x_3} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$, $\tilde{G}^1, \tilde{G}^1_{x_3} \in \mathbf{H}^{1,1}_{\beta_{12}}(\mathcal{U}')$. Applying (4.6) we have (4.5) in general. □

The regularity of higher derivates in x_3 is given in the next theorem.

Theorem 4.2 Suppose that $u \in \mathbf{H}^1(\Omega)$ be the weak solution of the problem (3.1) with $G^\ell \in \mathbf{H}^{2-\ell,2-\ell}_\beta(\Omega), \ell = 0,1$ and $f \in \mathbf{L}_\beta(\Omega)$.

(B1) If $f_{x_3^m} \in \mathbf{L}^2(\mathcal{U}'), G^\ell_{x_3^m} \in \mathbf{H}^{2-\ell}(\mathcal{U}'), \ell = 0,1\ ,\ 0 \le m \le k$, then $u_{x_3^{k+1}} \in \mathbf{H}^1(\mathcal{U})$, and

(4.18)

$$\left\|u_{x_3^{k+1}}\right\|_{\mathbf{H}^1(\mathcal{U})} \le C(k)\left\{\sum_{m=0}^{k}\left(\left\|f_{x_3^m}\right\|_{\mathbf{L}^2(\mathcal{U}')} + \sum_{\ell=0,1}\left\|G^\ell_{x_3^m}\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}')}\right) + \|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}')}\right\}.$$

(B2) If $f_{x_3^m} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}'), G^\ell_{x_3^m} \in \mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}'), \ell = 0,1\ ,\ 0 \le m \le k+1$, then $u_{x_3^{k+1}} \in \mathbf{H}^1(\mathcal{U})$, and

(4.19)

$$\left\|u_{x_3^{k+1}}\right\|_{\mathbf{H}^1(\mathcal{U})} \le C(k)\left\{\sum_{m=0}^{k}\left(\left\|f_{x_3^m}\right\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} + \sum_{\ell=0,1}\left\|G^\ell_{x_3^m}\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}')}\right) + \|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}')}\right\}.$$

Furthermore, if $f \in \mathbf{B}^0_{\beta_{12}}(\mathcal{U}')$ and $G^\ell \in \mathbf{B}^\ell_{\beta_{12}}(\mathcal{U}'), \ell = 0,1$ then

$$(4.20)\qquad \left\|u_{x_3^{k+1}}\right\|_{\mathbf{H}^1(\mathcal{U})} \le c d_3^{k+2}(k+2)! \quad \forall k \ge 0.$$

Proof. Let $\delta_\ell = \delta - \frac{\ell}{k}\Delta\delta$ and $\varepsilon_\ell = \varepsilon + \frac{\ell}{k}\Delta\varepsilon, \quad 0 \le \ell \le k$, where $\Delta\delta = \delta - \delta'$ and $\Delta\varepsilon = \varepsilon' - \varepsilon$. By $\mathcal{U}_\ell$ we denote $\mathcal{U}_{\varepsilon_\ell,\delta_\ell}$. Then $\mathcal{U} = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \cdots \subset \mathcal{U}_k = \mathcal{U}'$.

If the condition (B1) holds, the application of (4.4) leads to

$$\|u_{x_3}\|_{\mathbf{H}^1(\mathcal{U}_{k-1})} \le C_0 \left(\|f\|_{\mathbf{L}^2(\mathcal{U}_k)} + \sum_{\ell=0,1} \left\|G^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}_k)} + M(k)\|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_k)} \right),$$

$$\begin{aligned}
\left\|u_{x_3^2}\right\|_{\mathbf{H}^1(\mathcal{U}_{k-2})} &\le C_0 \left(\|f_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_{k-1})} + \sum_{\ell=0,1} \left\|G^\ell_{x_3}\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}_{k-1})} + M(k)\left\|u_{x_3^2}\right\|_{\mathbf{L}^2(\mathcal{U}_{k-1})} \right) \\
&\le C_0 \left(\|f_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_{k-1})} + \sum_{\ell=0,1} \left\|G^\ell_{x_3}\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}_{k-1})} + M(k)\left\|u_{x_3^2}\right\|_{\mathbf{L}^2(\mathcal{U}_{k-1})} \right) \\
&\le C_0^2 M(k) \left(\|f\|_{\mathbf{L}^2(\mathcal{U}_k)} + \sum_{\ell=0,1} \left\|G^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}_k)} \right) + C_0^2 M^2(k)\|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_k)}.
\end{aligned}$$

where $M(k) = kM_0 = k\max\left(\frac{1}{\Delta\varepsilon}, \frac{1}{\Delta\delta}\right)$. The argument above can be carried out for all $u_{x_3^m}, \quad 1 \le m \le k+1$. Hence $u_{x_3^m} \in \mathbf{H}^1(\mathcal{U}), 1 \le m \le k+1$, and by the mathematical induction it can be shown that for $0 \le s \le k$

$$\begin{aligned}
&\left\|u_{x_3^{s+1}}\right\|_{\mathbf{H}^1(\mathcal{U}_{k-s-1})} \\
&\le \sum_{m=0}^{s} C_0^{m+1} M^m(k) \left(\left\|f_{x_3^{s-m}}\right\|_{\mathbf{L}^2(\mathcal{U}_{k-s+m})} + \sum_{\ell=0,1} \left\|G^\ell_{x_3^{s-m}}\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}_{k-s+m})} \right. \\
&\left. + C_0^{s+1} M^{s+1}(k)\|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_k)} \right).
\end{aligned} \tag{4.21}$$

Then (4.18) follows from (4.21) immediately.

If the condition (B2) holds, we can analogously prove by mathematical induction that $u_{x_3^{s+1}} \in \mathbf{H}^1(\mathcal{U})$ for $0 \le s \le k$, and

(4.22)

$$\begin{aligned}
&\left\|u_{x_3^{s+1}}\right\|_{\mathbf{H}^1(\mathcal{U}_{k-s-1})} \\
\le\ & C_0 \left(\|f_{x^{s+1}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s})} + \sum_{\ell=0,1} \left\|G^\ell_{x_3^{s+1}}\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}_{k-s})} \right) \\
&+ \sum_{m=1}^{s} C_0^m M^m(k) \left\{ C_0 \left(\|f_{x_3^{s+1-m}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+m})} + \sum_{\ell=0,1} \|G^\ell_{x_3^{s+1-m}}\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}_{k-s+m})} \right) \right. \\
&\left. + \left\|f_{x_3^{s+1-m}}\right\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+m+1})} + \sum_{\ell=0,1} \left\|G^\ell_{x_3^{s+1-m}}\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}_{k-s+m+1})} \right\} \\
&+ C_0^{s+1} M^{s+1}(k) \left(\|f\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_k)} + \sum_{\ell=0,1} \left\|G^\ell\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}_k)} + \|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_k)} \right).
\end{aligned}$$

We shall prove (4.22) for $G^\ell = 0, \quad \ell = 0,1$. The proof for the case that $G^\ell \neq 0$ is similar to what follows. (4.22) holds for $s = 0$ due to (4.5) of Theorem 4.1. Suppose it is true up to s, then applying (4.5) to x_3^{s+1} we obtain

$$
\begin{aligned}
&\left\|u_{x_3^{s+2}}\right\|_{\mathbf{H}^1(\mathcal{U}_{k-s-2})} \\
&\leq C_0\left\{\|f_{x_3^{s+2}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s-1})}+M(k)(\|f_{x_3^{s+1}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s-1})}+\|u_{x_3^{s+1}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s-1})})\right\}.
\end{aligned}
$$

By the hypothesis of the induction we have

$$
\begin{aligned}
&\|u_{x_3^{s+2}}\|_{\mathbf{H}^1(\mathcal{U}_{k-s-2})} \\
&\leq C_0\|f_{x_3^{s+2}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s-1})}+C_0M(k)\|f_{x_3^{s+1}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s})}+C_0M(k)\{C_0\|f_{x_3^{s+1}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s})} \\
&\quad+\sum_{m=0}^{S} C_0^m M^m(k)(C_0\|f_{x_3^{s+1-m}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+m})}+\left\|f_{x_3^{s+1-m}}\right\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+m+1})}) \\
&\quad+C_0^{s+1}M^{s+1}(k)\left(\|f\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_k)}+\|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_k)}\right)\Big\} \\
&=C_0\left\|f_{x_3^{s+2}}\right\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s-1})}+\sum_{m=0}^{s+1} C_0^m M^m(k)\Big\{C_0\|f_{x_3^{s+2-m}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s-1+m})} \\
&\quad+\left\|f_{x_3^{s+2-m}}\right\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+m})}\Big\}+C_0^{s+2}M^{s+2}(k)\left(\|f\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_k)}+\|u_{x_3}\|_{\mathbf{L}^2(\mathcal{U}_k)}\right).
\end{aligned}
$$

Hence (4.22) holds for $(s+1)$, and then we complete the induction.

If $f\in\mathbf{B}^0_\beta(\mathcal{U}')$ and $G^\ell\in\mathbf{B}^{2-\ell}_\beta(\mathcal{U}')$, then there are some $d_3\geq 1$ and $C_3\geq 1$ such that for $k\geq 0$

$$
\|f_{x_3^k}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}\leq C_3\tilde{d}_3^k k!, \tag{4.23}
$$

$$
\left\|G^\ell_{x_3^k}\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}')}\leq C_3\tilde{d}_3^{k+2-\ell}(k+2-\ell)!.
$$

Substituting (4.23) into (4.22) and noting that $M(k)=kM_0$ we obtain

(4.24)

$$
\begin{aligned}
\|u_{x_3^{k+1}}\|_{\mathbf{H}^1(\mathcal{U})}\leq\ & \tilde{C}\{C_0C_3\tilde{d}_3^{k+1}(k+1)! \\
&+\sum_{m=1}^{k}(C_0+1)C_0^mM_0^m\tilde{d}_3^{k+1-m}(k+1-m)!k^m+C_0^{k+1}M_0^{k+1}k^{k+1}\}.
\end{aligned}
$$

By using Sterling formula: $k!=k^ke^{-k}\sqrt{2\pi k}\left(1+O\left(\frac{1}{k}\right)\right)$, we have for $1\leq m\leq k$

$$
\begin{aligned}
k^m(k+1-m)! &\leq Ck^m(k+1-m)^{k+1-m}e^{-(k+1-m)}\sqrt{2\pi(k+1-m)} \\
&\leq C(k+1)^{k+1}e^{-(k+1)}\sqrt{2\pi(k+1)}\cdot e^m \\
&\leq C(k+1)!e^m.
\end{aligned} \tag{4.25}
$$

We have by substituting (4.25) into (4.24)

$$
\begin{aligned}
\|u_{x_3^{k+1}}\|_{\mathbf{H}^1(\mathcal{U})} &\le \tilde{C}\Big\{C_0 C_3 \tilde{d}_3^{k+2}(k+2)! + \sum_{m=1}^{k}(C_0+1)(C_0M_0e)^m {d_3}^{k+1-m}(k+1)! \\
&\quad +(C_0M_0e)^{k+1}(k+1)!\Big\} \\
&\le \tilde{C} d_3^{k+2}(k+2)!.
\end{aligned}
$$

where $d_3 = \max\left(\tilde{d}_3, C_0M_0e\right)$. □

4.2 Regularity of high-order derivatives with respect to the direction perpendicular to the edges

We now turn our attention to the regularity of high-order derivates with respect the variables other than x_3. Let $\varphi_1(x_3)$ and $\varphi_2(r)$ be the C^∞ cut-off functions defined in (4.13), and let $v(x) = \varphi_1(x_3)\varphi_2(r)u(x)$ where $u(x)$ is the weak solution of (3.1). Then v satisfies

$$
(4.26) \qquad \begin{cases} -\Delta v = \tilde{f} & \text{in } \mathcal{U}' = \mathcal{U}_{\varepsilon',\delta'} \\ v\,|_{r=\varepsilon'} = v|_{x_3=\pm(1-\delta')} = 0, & \\ v\,\Big|_{\theta=0} = \tilde{g}^0 = \tilde{G}^0\Big|_{\theta=0}, & \dfrac{\partial v}{\partial n} = \tilde{g}^1 = \tilde{G}^1\,|_{\theta=\omega_{12}} \end{cases}
$$

where $\tilde{g}^\ell = \varphi_1(x_3)\varphi_2(r)\,g^\ell, \tilde{G}^\ell = \varphi_1(x_3)\varphi_2(r)\,G^\ell$ and $\tilde{f} = \varphi_1(x_3)\varphi_2(r)f + h$,

$$
\begin{aligned}
h = \;& 2(\nabla_{12}u\cdot\nabla_{12}\varphi_2(r))\varphi_1(x_3) + u\varphi_1(x_3)\Delta_{12}\varphi_2(r) \\
&+2u_{x_3}\varphi_1'(x_3)\varphi_2(r) + u\varphi_1''(x_3)\varphi_2(r)
\end{aligned}
$$

with $\nabla_{12} = (\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2})$ and $\Delta_{12} = (\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2})$. Obviously $\tilde{f} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$, $\tilde{G}^\ell \in \mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(\mathcal{U}'), \ell = 0,1$. Furthermore $v, \tilde{f}$ and $\tilde{G}^\ell$ vanish for $r > \varepsilon'$ or $|x_3| > 1-\delta'$, and with the constant M_0 given in (4.7).

$$
(4.27\text{a}) \quad \left\|\tilde{f}\right\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} \le C\left(\|f\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} + M_0\|u_r\|_{\mathbf{L}^2(\mathcal{U}')} + \sum_{s=0,1} M_0^{2-s}\left\|u_{x_3^s}\right\|_{\mathbf{L}^2(\mathcal{U}')}\right);
$$

$$
(4.27\text{b}) \qquad \begin{aligned}
\left\|\tilde{G}^0\right\|_{\mathbf{H}^{2,2}_{\beta_{12}}(\mathcal{U}')} &\le C\left(\sum_{s=0,1} M_0^{2-s}\,|G^0|_{\mathbf{H}^m(\mathcal{U}')} + |G^0|_{\mathbf{H}^{2,2}_{\beta_{12}}(\mathcal{U}')}\right) \\
&\le CM_0^{2-s}\,\|G^0\|_{H^{2,2}_{\beta_{12}}(\mathcal{U}')};
\end{aligned}
$$

$$
(4.27\text{c}) \qquad \begin{aligned}
\left\|\tilde{G}^1\right\|_{\mathbf{H}^{1,1}_{\beta_{12}}(\mathcal{U}')} &\le C\left(M_0\,\|G^1\|_{\mathbf{L}^2(\mathcal{U}')} + |G^1|_{\mathbf{H}^{1,1}_{\beta_{12}}(\mathcal{U}')}\right) \\
&\le CM_0\,\|G^1\|_{\mathbf{H}^{1,1}_{\beta_{12}}(\mathcal{U}')}.
\end{aligned}
$$

If $f \in \mathbf{L}^2(\mathcal{U}')$ and $G^\ell \in \mathbf{H}^{2-\ell}(\mathcal{U}')$, then $\tilde{f} \in \mathbf{L}^2(\mathcal{U}'), \tilde{G}^\ell \in \mathbf{H}^{2-\ell}(\mathcal{U}')$, and

$$(4.28a) \qquad \|\tilde{f}\|_{\mathbf{L}^2(\mathcal{U}')} \leq C\left(\|f\|_{\mathbf{L}^2(\mathcal{U}')} + M_0\|u_r\|_{\mathbf{L}^2(\mathcal{U}')} + \sum_{s=0,1} M_0^{2-s}\left\|u_{x_3^m}\right\|_{\mathbf{L}^2(\mathcal{U}')}\right),$$

$$(4.28b) \qquad \begin{aligned} \left\|\tilde{G}^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}')} &\leq C\sum_{s=0}^{2-\ell} M_0^{2-\ell-s}\left|G^\ell\right|_{\mathbf{H}^m(\mathcal{U}')} \\ &\leq CM_0^{2-\ell}\left\|G^\ell\right\|_{\mathbf{H}^{2-\ell}(\mathcal{U}')}. \end{aligned}$$

Let

$$\tilde{\mathbf{H}}^1(\mathcal{U}') = \left\{u \in \mathbf{H}^1(\mathcal{U}') \;\mid\; u|_{r=\epsilon'} = u|_{x_3=\pm(1-\delta')} = 0\right\}$$

and

$$\tilde{\mathbf{H}}_D^1(\mathcal{U}') = \left\{u \in \tilde{\mathbf{H}}^1(\mathcal{U}') \;\mid\; u|_{\theta=0} = 0\right\}.$$

Then $v - \tilde{G}^0 \in \tilde{\mathbf{H}}_D^1(\mathcal{U}')$ and satisfies the following variational equations

$$(4.29) \qquad \int_{\mathcal{U}'} \nabla v \cdot \nabla w dx = \int_{\mathcal{U}'} \tilde{f} dx + \int_{\Gamma_2 \bigcap \partial\mathcal{U}'} \tilde{g}^1 w dS, \quad \forall w \in \tilde{\mathbf{H}}_D^1(\mathcal{U}').$$

We now extend $v, \tilde{f}$ and $\tilde{G}^\ell$ into $Q'_\varepsilon \times \mathbb{R}^1$ by zero extension outside $\mathcal{U}'$. Then for almost every $\tilde{x} = (x_1, x_2) \in Q_{\epsilon'}$, $v(\tilde{x}, \cdot) \in H^1(\mathbb{R}^1)$, $\tilde{f}(\tilde{x}, \cdot) \in L^2(\mathbb{R}^1)$, $G'^\ell(\tilde{x}, \cdot) \in H^{2-\ell}(\mathbb{R}^1)$, $\ell = 0, 1$. Let $\mathcal{F}$ denote the Fourier transform, namely, for admissable function w

$$\tilde{w}(x, \lambda) = \mathcal{F}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x_1, x_2, x_3) e^{-ix_3\lambda} dx_3, \quad \lambda \in (-\infty, \infty).$$

Then $\hat{v} = \mathcal{F}(v), \hat{f} = \mathcal{F}(\tilde{f}), \hat{G}^\ell = \mathcal{F}(\tilde{G}^\ell), \ell = 0, 1$ exist and $\hat{v}$ solves the following problem

$$(4.30) \qquad \begin{cases} -\Delta_1 \hat{v} + \lambda^2 \hat{v} = \hat{f}, \text{ in } Q_{\epsilon'} = Q', \\ \hat{v}|_{r=\epsilon'} = 0, \\ \hat{v}|_{\theta=0} = \hat{g}^0 = \hat{G}^0|_{\theta=0}, \quad \dfrac{\partial \hat{v}}{\partial n}|_{\theta=\omega} = \hat{g}^1 = \hat{G}^1|_{\theta=\omega}. \end{cases}$$

Let $\mathbf{H}_D^1(Q') = \{\hat{w} \in H^1(Q') \mid \hat{w}|_{\theta=0} = 0\}$. Then $\hat{v} - \hat{G}^0 \in \mathbf{H}_D^1(Q')$ and satisfies the variational equation

$$(4.31) \qquad \int_{Q'} \left(\nabla_1 \hat{v} \cdot \nabla_1 \bar{\hat{w}} + |\lambda|^2 \hat{v}\bar{\hat{w}}\right) d\tilde{x} = \int_{Q'} \hat{f}\bar{\hat{w}} d\tilde{x} + \int_{\gamma_2} \hat{g}^1 \bar{\hat{w}} ds, \forall \hat{w} \in \mathbf{H}_D^1(Q')$$

where $\gamma_2 = \Gamma_2 \bigcap \partial Q'$.

We shall introduce the weighted Sobolev space $\mathbf{H}^{k,\ell}_{\beta_{12}}(Q')$ defined in []. For integer k and ℓ, $k \geq \ell \geq 0$, $\mathbf{H}^{k,\ell}_{\beta_{12}}(Q')$ is the completion of $C^\infty-$ function in the norm

$$\|\hat{u}\|^2_{\mathbf{H}^{k,\ell}_{\beta_{12}}(Q')} = \sum_{|\alpha'|=0}^{k} \left\|\Phi^{\alpha',\ell}_{\beta_{12}}(\tilde{x})\, D^{\alpha'}\hat{u}\right\|^2_{\mathbf{L}^2(Q')} \tag{4.32}$$

with $r(\tilde{x}) = |\tilde{x}| = (x_1^2 + x_2^2)^{1/2}$ and

$$\Phi^{\alpha',\ell}_{\beta_{12}}(\tilde{x}) = \begin{cases} r(\tilde{x})^{\beta_{12}+|\alpha'|-\ell}, & \text{for } |\alpha'| = \alpha_1 + \alpha_2 \geq \ell, \\ 1, & \text{for } |\alpha'| < \ell. \end{cases}$$

As usual we shall write $\mathbf{H}^{0,0}_{\beta_{12}}(Q') = \mathbf{L}_{\beta_{12}}(Q')$.

Lemma 4.2 Let $u \in \mathbf{H}^1(\Omega)$ be the weak solution of the problem (3.1) with $f \in \mathbf{L}_\beta(\Omega)$ and $G^\ell \in \mathbf{H}^{2-\ell,2-\ell}_\beta(\Omega), \ell = 0, 1$. If $\beta_{12} \in (0,1)$ satisfies

$$\beta_{12} > 1 - \kappa_{12}, \quad \kappa_{12} = \frac{\pi}{2w_{12}} \tag{4.33}$$

Then the weak solution $\hat{v}$ of the problem (4.30) belongs to $\mathbf{H}^{2,2}_{\beta_{12}}(Q')$ and

$$\|\hat{v}\|^2_{\mathbf{H}^{2,2}_{\beta_{12}}(Q')} \leq C\left(\left\|\hat{f}\right\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \sum_{\ell=0,1}\left\|\hat{G}^\ell\right\|^2_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(Q')} + |\lambda|^4\|\hat{v}\|^2_{\mathbf{L}^2(Q')}\right) \tag{4.34}$$

and

$$\begin{aligned} &\|\hat{v}\|^2_{\mathbf{H}^{2,2}_{\beta_{12}}(Q')} + |\lambda|^2\|\nabla_{12}\hat{v}\|^2_{\mathbf{L}^2(Q')} + |\lambda|^4\|\hat{v}\|^2_{\mathbf{L}^2(Q')} \\ &\leq C\left(1+|\lambda|^2\right)\left\{\left\|\hat{f}\right\|_{\mathbf{L}^2_{\beta_{12}}(Q')} + \sum_{\ell=0,1}\left\|\hat{G}^\ell\right\|_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(Q')} + |\lambda|^4\left\|\hat{G}^0\right\|^2_{\mathbf{L}^2(Q')}\right\}. \end{aligned} \tag{4.35}$$

Proof. Since $v \in \mathbf{H}^1\left(Q' \times \mathbb{R}^1\right)$, $\tilde{f} \in \mathbf{L}_{\beta_{12}}\left(Q' \times \mathbb{R}^1\right)$ and $\tilde{G}^\ell \in \mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}\left(Q' \times \mathbb{R}^1\right)$, $\hat{v} \in \mathbf{H}^1(Q')$, $\hat{f} \in \mathbf{L}_{\beta_{12}}(Q')$ and $\hat{G}^\ell \in \mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(Q')$ for almost every $\lambda \in \mathbb{R}^1$.

Because $\hat{v}$ is the weak solution of the problem (4.30), by Theorem 2.1 and Remark of [2], $\hat{v} \in \mathbf{H}^{2,2}_{\beta_{12}}(Q')$, and

$$\|\hat{v}\|^2_{\mathbf{H}^{2,2}_{\beta_{12}}(Q')} \leq C\left(\left\|\hat{f} - \lambda^2\hat{v}\right\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \sum_{\ell=0,1}\left\|\hat{G}^\ell\right\|^2_{\mathbf{H}^{2-\ell,2-\ell}_{\beta_{12}}(Q')}\right)$$

with C independent of λ. This leads to (4.34) immediately.

We now assume that $\hat{G}^0 = 0$. By Lemma 2.10 and 2.11 of [2] we get for any $\hat{w} \in \mathbf{H}^1(Q')$

$$\left|\int_{Q'} \hat{f}\bar{\hat{w}}\,d\tilde{x}\right| \leq C\|\hat{f}\|_{\mathbf{L}_{\beta_{12}}(Q')}\|\hat{w}\|_{\mathbf{H}^1(Q')},$$

and

$$|\int_{\gamma_2} \hat{g}^1 \bar{\hat{w}} ds| \le C \left\| \hat{G}^1 \right\|_{\mathbf{H}^{1,1}_{\beta_{12}}(Q')} \|\hat{w}\|_{\mathbf{H}^1(Q')}.$$

Letting $\hat{w} = \hat{v}$ and substituting the inequalities above into the variational equation (4.30) we obtain

$$\|\nabla_1 \hat{v}\|_{\mathbf{L}^2(Q')} \le C \left(\left\| \hat{f} \right\|_{\mathbf{L}_{\beta_{12}}(Q')} + \left\| \hat{G}^1 \right\|_{\mathbf{H}^{1,1}_{\beta_{12}}(Q')} \right) \tag{4.36a}$$

and

$$|\lambda| \, \|\hat{v}\|_{\mathbf{L}^2(Q')} \le C \left(\left\| \hat{f} \right\|_{\mathbf{L}_{\beta_{12}}(Q')} + \left\| \hat{G}^1 \right\|_{\mathbf{H}^{1,1}_{\beta_{12}}(Q')} \right). \tag{4.36b}$$

Here we used the inequality

$$\|\hat{v}\|_{\mathbf{L}^2(Q')} \le C \, \|\nabla_1 \hat{v}\|_{\mathbf{L}^2(Q')}, \quad \forall \hat{v} \in H^1_D(Q').$$

The combination of (4.34) and (4.36) leads to

$$\begin{aligned} &\|\hat{v}\|^2_{\mathbf{H}^{2,2}_{\beta_{12}}(Q')} + |\lambda|^2 \, \|\nabla_1 \hat{v}\|^2_{\mathbf{L}^2(Q')} + |\lambda|^4 \, \|\hat{v}\|^2_{\mathbf{L}^2(Q')} \\ &\le C \left(1 + |\lambda|^2\right) \left(\left\| \hat{f} \right\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \left\| \hat{G}^1 \right\|^2_{\mathbf{H}^{1,1}_{\beta_{12}}(Q')} \right). \end{aligned} \tag{4.37}$$

If $\hat{G}^0 \ne 0$ and $\hat{G}^1 = \hat{f} = 0$, setting $\hat{w} = \hat{v} - \hat{G}^0$ and applying the result above we get

$$\begin{aligned} &\|\hat{w}\|^2_{\mathbf{H}^{2,2}_{\beta_{12}}(Q')} + |\lambda|^2 \, \|\nabla_1 \hat{w}\|_{\mathbf{L}^2(Q')} + |\lambda|^4 \, \|\hat{w}\|_{\mathbf{L}^2(Q')} \\ &\le C(1 + |\lambda|^2) \left\{ |\lambda|^4 \|\hat{G}^0\|^2_{\mathbf{L}^2(Q')} + \sum_{\ell=0,1} \|\hat{G}^0\|^{2-\ell,2-\ell}_{\mathbf{H}_{\beta_{12}}(Q')} \right\}, \end{aligned}$$

which together with (4.37) implies (4.35) and then completes the lemma. □

Theorem 4.3. Let $u \in \mathbf{H}^1(\Omega)$ be the weak solution of the problem (3.1) with $f \in \mathbf{L}_\beta(\Omega)$ and $G^l \in \mathbf{H}^{2-l,2-l}_\beta(\Omega)$, $l = 0, 1$. In addition, $f_{x_3} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$ and $G^l_{x_3} \in \mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')$, $l = 0, 1$ with β_{12} satisfying (4.33). Then $u \in \mathbf{H}^{2,2}_{\beta_{12}}(\mathcal{U})$, and

$$\begin{aligned} \|u\|_{\mathbf{H}^{2,2}_{\beta_{12}}(\mathcal{U})} &\le C \sum_{m=0,1} \Big\{ \|f_{x_3^m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} + \sum_{l=0,1} |G^l_{x_3^m}|_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}} \\ &\quad + M_0 \|G^1\|_{\mathbf{L}^2(\mathcal{U}')} + \sum_{s=0,1} M_0^{2-s} |G^0|_{\mathbf{H}^s(\mathcal{U}')} \\ &\quad + M_0 \|u_{r x_3^m}\|_{\mathbf{L}^2(\mathcal{U}')} + \sum_{s=0}^{2} M_0^{2-s} \|u_{x_3^{s+m}}\|_{\mathbf{L}^2(\mathcal{U}')} \Big\} \\ &\le C \Big\{ \|f\|_{\mathbf{L}_{\beta_{12}}(\Omega)} + \|f_{x_3}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} \\ &\quad + \sum_{l=0,1} (\|G^l\|_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\Omega)} + \|G^l_{x_3}\|_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')}) \Big\}, \end{aligned} \tag{4.38}$$

$$
\begin{aligned}
&\sum_{|\alpha|=|\alpha'|=2} \|\Phi^{\alpha,2}_{\beta_{12}} D^\alpha u\|_{\mathbf{L}^2(\mathcal{U})} \\
(4.39)\qquad &\leq C\Big\{ \|f\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} + \sum_{l=0,1} |G^l|_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')} + M_0\|G^1\|_{\mathbf{L}^2(\mathcal{U}')} \\
&\quad + \sum_{l=0,1} M_0^{2-s}|G^0|_{\mathbf{H}^s(\mathcal{U}')} + M_0\|u_r\|_{\mathbf{L}^2(\mathcal{U}')} + \sum_{s=0}^{2} M_0^{2-s}\|u_{x_3^s}\|_{\mathbf{L}^2(\mathcal{U}')}\Big\},
\end{aligned}
$$

where M_0 is a constant given by (4.7).

Proof. Since $f \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$ and $G^l \in \mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')$, $\tilde{f} \in \mathbf{L}_{\beta_{12}}(Q' \times \mathbb{R}^1)$ and $\tilde{G}^l \in \mathbf{H}^{2-l,2-l}_{\beta_{12}}(Q' \times \mathbb{R}^1)$, $l = 0,1$. Hence $\hat{f} \in \mathbf{L}_{\beta_{12}}(Q')$ and $\hat{G}^l \in \mathbf{H}^{2-l,2-l}_{\beta_{12}}(Q')$ for almost every $\lambda \in \mathbb{R}^1$, and

$$
(4.40a)\qquad \int_{-\infty}^{\infty} \|\hat{f}\|^2_{\mathbf{L}_{\beta_{12}}(Q')} d\lambda = \|\tilde{f}\|^2_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')},
$$

and for $l = 0,1$

$$
(4.40b)\qquad \int_{-\infty}^{\infty} \|\hat{G}^l\|^2_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(Q')} d\lambda = \|\tilde{G}^l\|^2_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')}.
$$

By Lemma 4.2, $\hat{v} = \mathcal{F}(v) \in \mathbf{H}^{2-l,2-l}_{\beta_{12}}(Q')$, and (4.34)-(4.35) hold. If $f_{x_3^m} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}')$ and $G^l_{x_3^m} \in \mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')$ for $m = 0,1$ and $l = 0,1$, due to (4.34), we have

$$
\begin{aligned}
&\sum_{|\alpha|=|\alpha'|=2} \|\Phi^{\alpha,2}_{\beta_{12}} D^\alpha v\|^2_{\mathbf{L}^2(Q'\times\mathbb{R}^1)} \\
&\leq C\int_{-\infty}^{\infty} (\|\hat{f}\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \sum_{l=0,1} \|\hat{G}^l\|^2_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(Q')} + |\lambda|^4\|\hat{v}\|^2_{\mathbf{L}^2(Q')}) d\lambda
\end{aligned}
$$

by (4.40)

$$
\leq C\{\|\tilde{f}\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \sum_{l=0,1} |\hat{G}^l|^2_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')} + \|v_{x_3^2}\|^2_{\mathbf{L}^2(\mathcal{U}')}\}
$$

by (4.27)

$$
\begin{aligned}
&\leq C\{\|f\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \sum_{l=0,1} |G^l|^2_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')} + M_0^2\|G^1\|^2_{\mathbf{L}^2(\mathcal{U}')} \\
&\quad + \sum_{s=0,1} M_0^{2(2-s)}|G^0|^2_{\mathbf{H}^s(\mathcal{U}')} + \sum_{s=0}^{2} M_0^{2(2-s)}\|u_{x_3^s}\|^2_{\mathbf{L}^2(\mathcal{U}')} + M_0^2\|u_r\|^2_{\mathbf{L}^2(\mathcal{U}')}\}.
\end{aligned}
$$

This leads immediately to (4.39).

Similarly, due to (4.35), we have

$$
\begin{aligned}
&\|u\|^2_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U})} \leq \|v\|^2_{\mathbf{H}^{2,2}_{\beta_{12}}(\mathcal{U}')} \\
&\leq C\int_{-\infty}^{\infty} (\|\hat{v}\|^2_{\mathbf{H}^{2,2}_{\beta_{12}}(\mathcal{U}')} + |\lambda|^2\|\nabla_{12}\hat{v}\|^2_{\mathbf{L}^2(Q')} + |\lambda|^4\|\hat{v}\|^2_{\mathbf{L}^2(Q')}) d\lambda \\
(4.41)\qquad &\leq C\int_{-\infty}^{\infty} (1+|\lambda|^2)(\|\hat{f}\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \sum_{l=0,1} \|\hat{G}^l\|^2_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(Q')} + |\lambda|^4\|\hat{G}^0\|^2_{\mathbf{L}^2(Q')}) d\lambda \\
&\leq C\sum_{m=0,1} (\|\tilde{f}_{x_3^m}\|^2_{\mathbf{L}_{\beta_{12}}(Q')} + \sum_{l=0,1} \|\tilde{G}^l_{x_3^m}\|_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(Q')}).
\end{aligned}
$$

where $\tilde{f}_{x_3} = \phi_1(x_3)\phi_2(r)f_{x_3}$ and $\tilde{G}^l_{x_3} = \phi_1(x_3)\phi_2(r)G^l_{x_3}$. Therefore we have estimates of $\tilde{f}_{x_3}$ and $\tilde{G}^l_{x_3}$, which are similar to those in (4.27) except that $\tilde{f}$, $\tilde{G}^l, u_r$ and $u_{x_3^s}$ are replaced by $\tilde{f}_{x_3}, \tilde{G}^l_{x_3}, u_{rx_3}$ and $u_{x_3^{s+1}}$. Combining these with (4.27) and (4.41), we obtain (4.38). □

Remark 4.1 Although there are similarities between the regularity of solutions in the edge neighborhoods and those of solutions for the problems in polygonal domains in $\mathbb{R}^2$, there are substantial differences. For the problems on a polygonal domain Ω in $\mathbb{R}^2$, the boundary values problem of Poisson equation realizes an isomorphism $\mathbf{H}^{k+2,2}_\beta(\Omega) \to \mathbf{H}^{k,0}_\beta(\Omega) \times \mathbf{H}^{k+3/2,3/2}_\beta(\Gamma^0) \times \mathbf{H}^{k+1/2,1/2}_\beta(\Gamma^1), k \geq 0$, but it is not true for the problem on a polyhedral domain Ω in $\mathbb{R}^3$, namely, the conditions that $f \in \mathbf{H}^{k+2,2}_\beta(\Omega), G^l \in \mathbf{H}^{k+2-l,2-l}_\beta(\Gamma^l), l = 0,1$ are not sufficient to guarrantee the solution $u \in \mathbf{H}^{k+2,2}_{\beta_{12}}(\mathcal{U})$. □

Theorem 4.4 Let $f(x) \in \mathbf{L}_\beta(\Omega)$ and $G^l \in \mathbf{H}^{2-l,2-l}_\beta(\Omega), l = 0,1$. If $f_{x_3^m} \in \mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}')$ and $G^l_{x_3^m} \in \mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')$ for $m = 0,1$ and $k \geq 0$ with β_{12} satisfying (4.33), then the problem (3.1) has a unique solution (weak) $u \in \mathbf{H}^1(\Omega) \cap \mathbf{H}^{k+2,2}_{\beta_{12}}(\mathcal{U})$, and for $|\alpha| \leq k+2$

(4.42)
$$\begin{aligned}\|\Phi^{\alpha,2}_{\beta_{12}} r^{-\alpha_2}\mathcal{D}^\alpha u\|_{\mathbf{L}^2(\mathcal{U})} \leq C\Big\{&\sum_{m=0,1}(\|f_{x_3^m}\|_{\mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}')} + \sum_{l=0,1}\|G^l_{x_3^m}\|_{\mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')}) \\ &+\|f\|_{\mathbf{L}_\beta(\Omega)} + \sum_{l=0,1}\|G^l\|_{\mathbf{H}^{2-l,2-l}_\beta(\Omega)}\Big\}.\end{aligned}$$

Furthermore, if $f \in \mathbf{B}^0_{\beta_{12}}(\mathcal{U}')$ and $G^l \in \mathbf{B}^{2-l}_{\beta_{12}}(\mathcal{U}'), l = 0,1$, then $u \in \mathbf{B}^2_{\beta_{12}}(\mathcal{U})$, and there are some constants $C \geq 1$ and $d_i \geq 1$ such that for all α

$$\|\Phi^{\alpha,l}_{\beta_{12}} r^{-\alpha_2}\mathcal{D}^\alpha u\|_{\mathbf{L}^2(\mathcal{U})} \leq C d^\alpha \alpha!. \tag{4.43}$$

Proof. The assertion for $|\alpha| = 2$ is ture owing to Theorem 4.3. Let $\delta^* = \frac{1}{2}(\delta + \delta')$ and $\epsilon^* = \frac{1}{2}(\delta + \delta')$. (4.42) and (4.43) hold over $\mathcal{U}^* = \mathcal{U}_{\epsilon^*,\delta^*}$ for $|\alpha| = k > 2$ and $\alpha_1 + \alpha_2 \leq 2$ due to Theorem 4.2. It remains to show (4.42) and (4.43) for α with $\alpha_1 + \alpha_2 > 2$. To this end, set $\mathcal{U}_i = \mathcal{U}_{\epsilon_i,\delta_i}, 0 \leq i \leq k$, with $\epsilon_i = \epsilon + i\dfrac{\epsilon^* - \epsilon}{k}$, $\delta_i = \delta - i\dfrac{\delta - \delta^*}{k}$, so that $\mathcal{U}_0 = \mathcal{U}_{\epsilon,\delta} = \mathcal{U}$ and $\mathcal{U}_k = \mathcal{U}_{\epsilon^*,\delta^*} = \mathcal{U}^*$. Let $v = r^s u_{r^s x_3^t}, s + t \leq k$, $s, t \geq 0$. Then v satisfies

$$\begin{cases} -\Delta v = f_{s,t} = r^{s-2}(r^2 f_{x_3^t})_{r^s}, & \text{in } \mathcal{U}' \\ v|_{\theta=0} = r^s G^0_{r^s x_3^t}|_{\theta=0} = G^0_{s,t}|_{\theta=0}, & \\ \dfrac{\partial v}{\partial n}|_{\theta=\omega_{12}} = r^{s-1}(rG^1_{x_3^t})_{r^s}|_{\theta=\omega_{12}} = G^1_{s,t}|_{\theta=\omega_{12}}. & \end{cases} \tag{4.44}$$

Obviously $f_{s,t} \in \mathbf{L}_{\beta_{12}}(\mathcal{U}'), G^l_{s,t} \in \mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}'), l = 0, 1$ and

$$\|f_{s,t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} \leq C\|f\|_{\mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}')},$$

$$\|G^l_{s,t}\|_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}')} \leq C\|G^l\|_{\mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')}.$$

Due to Theorem 4.2, we have for $t = k$ and $s = 0$

$$\begin{aligned} \|r^{\beta_{12}+s} u_{r^{s+2}x_3^t}\|_{\mathbf{L}^2(\mathcal{U}_{k-s})} &\leq C(\|f_{x_3}\|_{\mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}')} + \sum_{l=0,1} \|G^l_{x_3^m}\|_{\mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')} \\ &\quad + \|f\|_{\mathbf{L}_\beta(\Omega)} + \sum_{l=0,1} \|G^l\|_{\mathbf{H}^{2-l,2-l}_\beta(\Omega)}). \end{aligned} \tag{4.45}$$

Suppose (4.45) holds up to $(s-1)$ with $0 \leq s + t \leq k$. Then the application of (4.39) of Theorem 4.3 to the problem (4.44) gives us

$$\begin{aligned} &\sum_{|\alpha|=|\alpha'|=2} \|\Phi^{\alpha,2}_{\beta_{12}} r^{-\alpha_2} \mathcal{D}^\alpha (r^s u_{r^s x_3^t})\|_{\mathbf{L}^2(\mathcal{U}_{k-s})} \\ &\leq C\{\sum_{m=0,1} (\|f_{s,t+m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+1})} + \sum_{l=0,1} \|G^l_{s,t+m}\|_{\mathbf{H}^{2-l,2-l}_{\beta_{12}}(\mathcal{U}_{k-s+1})}) \\ &\quad + \|(r^s u_{r^s x_3^{t+m}})_r\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} + \sum_{m=0}^{2} \|r^s u_{r^s x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})}\}. \end{aligned} \tag{4.46}$$

By the assumption of the induction, we have for $m = 0, 1$

$$\begin{aligned} &\|(r^s u_{r^s x_3^{t+m}})_r\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} \\ &\leq C(\|r^{s-1+\beta_{12}} u_{r^{s+1}x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} + \|r^{s-2+\beta_{12}} u_{r^s x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+2})}) \\ &\leq C(\|f_{x_3}\|_{\mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}')} + \sum_{l=0,1} \|G^l_{x_3}\|_{\mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')} + \|f\|_{\mathbf{L}_\beta(\Omega)} \\ &\quad + \sum_{l=0,1} \|G^l\|_{\mathbf{H}^{2-l,2-l}_\beta(\Omega)}) \end{aligned} \tag{4.47}$$

and for $m = 0, 1, 2$

$$\begin{aligned} &\|r^s u_{r^s x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} \\ &\leq \|r^{s-2+\beta_{12}} u_{r^s x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+2})} \\ &\leq C(\|f_{x_3}\|_{\mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}')} + \sum_{l=0,1} \|G^l_{x_3}\|_{\mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')} + \|f\|_{\mathbf{L}_\beta(\Omega)} \\ &\quad + \sum_{l=0,1} \|G^l\|_{\mathbf{H}^{2-l,2-l}_\beta(\Omega)}) \end{aligned} \tag{4.48}$$

Combining (4.46)-(4.48) we obtain for $0 \leq s \leq k$ and $s + t \leq k$

$$\begin{aligned} &\sum_{|\alpha|=|\alpha'|=2} \|\Phi^{\alpha,2}_{\beta_{12}} r^{s-\alpha_2} \mathcal{D}^\alpha u_{r^s x_3^t}\|_{\mathbf{L}^2(\mathcal{U}_{k-s})} \\ &\leq C(\|f_{x_3}\|_{\mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}_{k-s})} + \|f\|_{\mathbf{L}_\beta(\Omega)} + \sum_{l=0,1} \|G^l_{x_3}\|_{\mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')} \\ &\quad + \|G^l\|_{\mathbf{H}^{2-l,2-l}_\beta(\Omega)}). \end{aligned} \tag{4.49}$$

This completes (4.42) by the induction for α with $|\alpha| = k+2$ and $\alpha_2 \le 2$.

Now it remains to show (4.42) for $\alpha_2 > 2$. For α with $|\alpha| = k+2$ and $\alpha_2 > 2$, let $w = r^{\alpha_1} u_{r^{\alpha_1}\theta^{\alpha_2-2}x_3^{\alpha_3}}$. Then

$$-\Delta w = r^{\alpha_1-2}(r^2 f_{\theta^{\alpha_2-2}x_3^{\alpha_3}})_{r^{\alpha_1}}.$$

Noting that

$$\begin{aligned}\Delta w = \; & r^{\alpha_1-2}\mathcal{D}^\alpha u + (2\alpha_1+1) r^{\alpha_1-1} u_{r^{\alpha_1+1}\theta^{\alpha_2-2}x_3^{\alpha_3}} \\ & + \alpha_1^2 r^{\alpha_1-2} u_{r^{\alpha_1}\theta^{\alpha_2-2}x_3^{\alpha_3}} + r^{\alpha_1} u_{r^{\alpha_1+2}\theta^{\alpha_2-2}x_3^{\alpha_3}} + r^{\alpha_1} u_{r^{\alpha_1}\theta^{\alpha_2-2}x_3^{\alpha_3+2}}.\end{aligned}$$

we obtain

$$\begin{aligned}& \|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} \\ & \le \|r^{\alpha_1-2}(r^2 f_{\theta^{\alpha_2-2}x_3^{\alpha_3}})_{r^{\alpha_1}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} + (2\alpha+1)\|r^{\alpha_1-1} u_{r^{\alpha_1+1}\theta^{\alpha_2-2}x_3^{\alpha_3}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} \\ & \quad + \alpha_1^2 \|r^{\alpha_1-2} u_{r^{\alpha_1}\theta^{\alpha_2-2}x_3^{\alpha_3}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} + \|r^{\alpha_1} u_{r^{\alpha_1+2}\theta^{\alpha_2-2}x_3^{\alpha_3}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} \\ & \quad + \|r^{\alpha_1} u_{r^{\alpha_1}\theta^{\alpha_2-2}x_3^{\alpha_3+2}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})}.\end{aligned} \tag{4.50}$$

Then simple induction over α_2 leads to

$$\begin{aligned}& \|r^{\alpha_1-2} u_{r^{\alpha_1}\theta^{\alpha_2}x_3^{\alpha_3}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} \\ & \le C(\|f_{x_3}\|_{\mathbf{H}^{k,0}_{\beta_{12}}(\mathcal{U}')} + \|f\|_{\mathbf{L}_\beta(\Omega)} + \sum_{l=0,1}(\|G^l_{x_3}\|_{\mathbf{H}^{k+2-l,2-l}_{\beta_{12}}(\mathcal{U}')} + \|G^l\|_{\mathbf{H}^{2-l,2-l}_{\beta}(\Omega)}).\end{aligned}$$

This completes (4.42).

We now shall prove (4.43). We assume that $G^l = 0, l = 0,1$ for simplicity, and we establish the following estimates by mathematical induction

$$\begin{aligned}\|r^s u_{r^{s+2}x_3^t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s})} & \le C_*\{\sum_{l=0}^{s}\sum_{m=0}^{l}\|f_{s-l,t+m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} D_1^l D_3^{-m} k^{l-m} \\ & \quad + \sum_{m=0}^{s}\|u_{rx_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^{s+1} D_3^{-m} k^{s-m+1} \\ & \quad + \sum_{m=0}^{s+2}\|u_{x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^{s+2} D_3^{-m} k^{s-m+2}\}\end{aligned} \tag{4.51}$$

where $0 \le s+t \le k, D_1$ and D_3 are suitable constants. For $s=0$, (4.49) holds due to (4.18) of Theorem 4.2. Suppose it is true up to $(s-1)$ with $s+t \le k$. Then by application of (4.39) of Theorem 4.3 to the equation (4.44), we have

$$\begin{aligned}& \sum_{|\alpha|=|\alpha'|=2}\|\Phi^{\alpha,2}_{\beta_{12}} r^{-\alpha_2}\mathcal{D}^\alpha(r^s u_{r^s x_3^t})\|_{\mathbf{L}^2(\mathcal{U}_{k-s})} \\ & \le C_0\{\|f_{s,t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+1})} + (M_* k)\|r^s u_{r^{s+1}x_3^t}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} \\ & \quad + M_* k^2 \|r^{s-1} u_{r^s x_3^t}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} + \sum_{t'=0,1}(M_* k)^{2-t'}\|r^s u_{r^s x_3^{t+t'}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})}\}\end{aligned} \tag{4.52}$$

where $M_* = \max\{\frac{2}{\Delta\delta}, \frac{2}{\Delta\epsilon}\} > 1$. By the hypothese of induction

$$
\begin{aligned}
&\|r^s u_{r^{s+1}x_3^t}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} \\
&\leq \|r^{s-1} u_{r^{s+1}x_3^t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+1})} \\
&\leq C_*\{\sum_{l=0}^{s-1}\sum_{m=0}^{l} \|f_{s-1-l,t+m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} D_1^l D_3^{-m} k^{l-m} \\
&+ \sum_{m=0}^{s-1} \|u_{rx_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_s^s D_3^{-m} k^{s-m} \\
&+ \sum_{m=0}^{s+1} \|u_{x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^{s+1} D_3^{-m} k^{s-m+1}\},
\end{aligned} \tag{4.53}
$$

$$
\|r^{s-1} u_{r^s x_3^t}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} \leq \|r^{s-2} u_{r^s x_3^t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+2})}, \tag{4.54a}
$$

and

$$
\begin{aligned}
&\|r^s u_{r^s x_3^{t+t'}}\|_{\mathbf{L}^2(\mathcal{U}_{k-s+1})} \\
\leq\ & \|r^{s-2} u_{r^s x_3^{t+t'}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}_{k-s+2})} \\
\leq\ & C_*\{\sum_{l=0}^{s-2}\sum_{m=0}^{l} \|f_{s-2-l,t+t'+m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} D_1^l D_3^{-m} k^{l-m} \\
&+ \sum_{m=0}^{s-2} \|u_{rx_3^{t+t'+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_s^{s-1} D_3^{-m} k^{s-1-m} \\
&+ \sum_{m=0}^{s} \|u_{x_3^{t+t'+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^s D_3^{-m} k^{s-m}\}
\end{aligned} \tag{4.54b}
$$

The combination of (4.52)-(4.54) leads to

$$
\|r^s u_{r^{s+2}x_3^t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} \leq A + B + E,
$$

$$
\begin{aligned}
A &= C_0\|f_{s,t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} + \sum_{l=0}^{s-1}\sum_{m=0}^{l}(2+M_*)C_*\|f_{s-1-l,t+m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} D_1^l D_3^{-m} k^{l-m+1} \\
&+C_* \sum_{t'=0}^{2}\sum_{l=0}^{s-2}\sum_{m=0}^{l} \|f_{s-2-l,t+t'+m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} D_1^l D_3^{-m} k^{t-m+2-t'}(M_*+1)^{2-t'} \\
&\leq C_* \sum_{l=0}^{s}\sum_{m=0}^{l} \|f_{s-l,t+m}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} D_1^l D_3^{-m} k^{l-m},
\end{aligned}
$$

$$
\begin{aligned}
B &= C_0 C_*\{\sum_{m=0}^{s-1} \|u_{rx_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^s D_3^{-m} k^{s-m+1} M_* \\
&+ \sum_{t'=0}^{2}\sum_{m=0}^{s-2} \|u_{rx_3^{t+m+t'}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^s D_3^{-m} k^{s-m+1-t'} M_*^{2-t'}\} \\
&\leq C_* \sum_{m=0}^{s} \|u_{rx_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^{s+1} D_3^{-m} k^{s-m+1}
\end{aligned}
$$

and

$$
\begin{aligned}
E &= C_0 C_*\{\sum_{m=0}^{s+1} \|u_{x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^{s+1} D_3^{-m} k^{s-m+2} M_* \\
&+ \sum_{t'=0}^{2}\sum_{m=0}^{s} \|u_{x_3^{t+m+t'}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^s D_3^{-m} k^{s-m+2-t'} M_*\} \\
&\leq C_* \sum_{m=0}^{s+2} \|u_{x_3^{t+m}}\|_{\mathbf{L}^2(\mathcal{U}^*)} D_1^{s+2} D_3^{-m} k^{s-m+2}
\end{aligned}
$$

with $C_* \geq C_0$ and $D_1 \geq 4C_* \max(2+M_*, d_3)$. This completes the induction.

If $f \in \mathbf{B}^0_{\beta_{12}(\mathcal{U}')}, G^l = 0, l = 0,1$ then by (4.19) of Theorem 4.2, we have for α with $|\alpha| = \alpha_1 + \alpha_2 \leq 2$,

$$\|\Phi^{\alpha,2}_{\beta_{12}} r^{-\alpha_2} \mathcal{D}^\alpha u\|_{\mathbf{L}^2(\mathcal{U}_*)} \leq C_1 D_3^{\alpha_3} \alpha_3! \tag{4.55a}$$

and we may assume for all α that

$$\|r^{\alpha_1 - 2}(r^2 f_{\theta^{\alpha_2} x_3^{\alpha_3}})_{r^{\alpha_1}}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} \leq C_1 D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} \alpha!. \tag{4.55b}$$

Substituting (4.55) into (4.51), we obtain for $s+t=k$

$$\begin{aligned}
&\|r^s u_{r^{s+2} x_3^t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} \\
\leq\ & C_*\{\sum_{l=0}^{s}\sum_{m=0}^{l} C_1 D_1^s D_3^t (s-l)!(t+m)! k^{l-m} \\
&+ \sum_{m=0}^{s} C_1 D_1^{s+1} D_3^t k^{s-m+1}(t+m)! + \sum_{m=0}^{s+2} C_1 D_1^{s+2} D_3^t k^{s-m+2}(t+m)!\} \\
\leq\ & C_*\{4C_1 D_1^s D_3^t t! k^s + 2C_1 D_1^{s+1} D_3^t t! k^{s+1} + 2C_1 D_1^{s+2} D_3^t t! k^{s+2}\} \\
\leq\ & C_2 D_1^{s+2} D_3^t k^{s+t+2}.
\end{aligned}$$

Note that by Sterling's formula

$$k^{s+l+2} = k^{k+2} \leq C_3 (k+2)! e^{k+2} \sqrt{2\pi(k+2)} \leq \tilde{C}_3 (k+2)!(2e)^{k+2}$$

which implies

$$\|r^s u_{r^{s+2} x_3^t}\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U}')} \leq C_4 D_1^{s+2} D_3^t (2e)^{k+2}(k+2)! \leq C_4 d_1^{s+2} d_3^t t!(s+2)!.$$

Hence (4.43) is proved for α with $\alpha_2 = 0, d_1 = 4eD_1$ and $d_3 = 4eD_3$. In the same manner (4.51) and (4.56) can be proved for α with $\alpha_2 \leq 2$ except that $v = r^{\alpha_1} u_{r^{\alpha_1} \theta^{\alpha_2} x_3^{\alpha_3}}$, $|\alpha| \leq k$ and $\alpha_2 \leq 2$ satisfies the equation (4.44) and the estimates (4.45), (4.51) and (4.56) instead of $v = r^s u_{r^s x_3^t}$ with $s+t \leq k$. It remains to prove (4.42) for α with $\alpha_2 > 2$. Suppose that (4.43) holds up to $(\alpha_2 - 1)$, then by (4.50) and (4.55) we obtain

$$\begin{aligned}
&\|r^{\alpha_1 - 2} D^\alpha u\|_{\mathbf{L}_{\beta_{12}}(\mathcal{U})} \\
&\leq C_1 D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} \alpha! + C_4\{(2\alpha+1) d_1^{\alpha_1 - 1} d_2^{\alpha_2 - 2} d_3^{\alpha_3}(\alpha_1+1)!(\alpha_2-2)!\alpha_3! \\
&\quad + \alpha_1^2 d_1^{\alpha_1} d_2^{\alpha_2 - 2} d_3^{\alpha_3} \alpha_1!(\alpha_2-2)!\alpha_3! + d_1^{\alpha_1+2} d_2^{\alpha_2-2} d_3^{\alpha_3}(\alpha_1+2)(\alpha_2-2)!\alpha_3! \\
&\quad + d_1^{\alpha_1} d_2^{\alpha_2-2} d_3^{\alpha_3+2} \alpha_1!(\alpha_1-2)!(\alpha_3+2)!\} \\
&\leq C_4 d^\alpha \alpha!
\end{aligned}$$

where $C_4 \geq C_1$ and $d_2 \geq 2\max(d_1, D_2, d_3)$. This completes the induction. Then (4.43) holds for all α, and $u \in \mathbf{B}^2_{\beta_{12}}(\mathcal{U})$.

Remark 4.2 If $f(x) \in \mathbf{L}_\beta(\Omega)$, $G^l \in \mathbf{H}_\beta^{2-l,2-l}(\Omega)$ with $\beta_{12}=0, l=0,1$, then (4.4) of Theorem 4.1 and (4.34) of Lemma 4.2 implies that $u(x) \in \mathbf{H}_{\tilde\beta_{12}}^{2,2}(\mathcal{U}')$ with $\tilde\beta_{12}$ satisfying (4.33), and

$$\|u\|_{\mathbf{H}_{\tilde\beta_{12}}^{2,2}(\mathcal{U})} \le C(\|f\|_{\mathbf{L}_\beta(\Omega)} + \sum_{l=0,1} \|G^l\|_{\mathbf{H}_\beta^{2-l,2-l}(\Omega)}).$$

Further, if $f(x) \in \mathbf{H}_{\beta_{12}}^{k,0}(\mathcal{U}') \cap \mathbf{L}_\beta(\Omega)$, $G^l \in \mathbf{H}_{\beta_{12}}^{k+2-l,2-l}(\mathcal{U}') \cap \mathbf{H}_\beta^{2-l,2-l}(\Omega)$ with $\beta_{12}=0$, then $u(x) \in \mathbf{H}_{\tilde\beta_{12}}^{k+2,2}(\mathcal{U})$ with $\tilde\beta_{12}$ satisfying (4.33), and

$$\|u\|_{\mathbf{H}_{\tilde\beta_{12}}^{k+2,2}(\mathcal{U})} \le C\{\|f\|_{\mathbf{L}_\beta(\Omega)} + \|f(x)\|_{\mathbf{H}_{\beta_{12}}^{k,0}(\mathcal{U}')} + \sum_{l=0,1} \|G^l\|_{\mathbf{H}_\beta^{2-l,2-l}(\Omega)} + \|G^l\|_{\mathbf{H}_{\beta_{12}}^{k+2-l,2-l}(\mathcal{U}')}\}.$$

$\square$

As a consequence of Theorem 4.4 and Theorem 2.1, we have regularity of solution in the countably normed space with weighted $\mathbf{C}^k-$ norms.

Theorem 4.5 If $f(x) \in \mathbf{B}_{\beta_{12}}^0(\mathcal{U}') \cap \mathbf{L}_\beta(\Omega)$, $G^l(x) \in \mathbf{B}_{\beta_{12}}^{2-l}(\mathcal{U}') \cap \mathbf{H}_\beta^{2-l,2-l}(\Omega)$, then the weak solution $u(x)$ of the Poisson equation (3.1) belongs to $\mathbf{C}_{\beta_{12}}^2(\mathcal{U}) \cap \mathbf{H}^1(\Omega)$. $\square$

Remark 4.3 In special case that the data functions are analytic or piecewise analytic, namely,

(i) function f is analytic in $\bar\Omega$,

(ii) $g^\ell, \ell = 0,1$, are analytic on every face $\bar\Gamma_i \subset \Gamma^0$ and $\bar\Gamma_j \subset \bar\Gamma^1$.

Then the solution u of the problem (3.1) belongs to $\mathbf{B}_{\beta_{12}}^2(\mathcal{U}) \cap \mathbf{H}^1(\Omega)$ and $\mathbf{C}_{\beta_{12}}^2(\mathcal{U}) \cap \mathbf{H}^1(\Omega)$. $\square$

Remark 4.4 The regularity described by the countably normed space with weighted $\mathbf{C}^k-$norm implies the pointwise estimates of the derivatives of solution of all orders, namely, for $x \in \bar{\mathcal{U}}_{ij}, |\alpha| = k \ge 0$,

$$|D^\alpha(u(x) - u(0,0,x_3))| \le C d^\alpha \alpha! r^{-(\beta_{ij}+|\alpha|-1)} \tag{4.57}$$

and

$$\left|\frac{d^k}{dx_3^k} u(0,0,x_3)\right| \le C d_3^k k!. \tag{4.58}$$

In many applications, for instance, the error analysis of the p and $h-p$ version of the finite or boundary element method, we prefer to use the pointwise estimates of the high order derivatives of solution instead of the weighted Sobolev norms of high order derivatives. By using estimates (4.57) and (4.58) we have shown in [8,9,16] that the approximation to functions belonging to $\mathbf{C}_{\beta_{12}}^2(\bar{\mathcal{U}})$ converges exponentially by properly designed piecewise polynomial spaces. $\square$

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